MULTIPLICATIVE LINEAR FUNCTIONALS
ON CONVOLUTION ALGEBRAS

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ABSTRACT. It is shown that semicharacters on the semigroup $S$ lead in a natural way to multiplicative linear functionals on $l(S)$, the convolution algebra of all complex valued functions on $S$. A theorem of D. H. Lehmer and a theorem of M. Tainiter follow as special cases.


2. Notation.

2.1. $S$ will always denote a commutative semigroup with an identity, $e$, (we could omit the identity but keep it for convenience). Also, for each $n \in S$ the set $\{(x, y) \in S \times S | xy = n\}$ is finite.

2.2. Let $I = \{x \in S | x^2 = x\}$ be the set of idempotent elements in $S$.

2.3. Let $A_n = \{x \in S | xn = n\}$ be the set of associates of $n$.

2.4. Let $D_n = \{x \in S | \exists y \in S$ and $xy = n\}$ be the set of divisors of $n$. None of the sets $I, A_n$, and $D_n$ is empty.

2.5. Let $l(S)$ be the set of all complex-valued functions on $S$. Then $l(S)$ is a convolution algebra [4] with convolution given by

$$f * g(n) = \sum_{xy = n} f(x)g(y).$$


3.1. Definition. Let $T$ be any semigroup. A semicharacter on $T$ is a bounded, multiplicative, complex-valued function $\chi$ on $T$ which is not identically zero. For our purposes we need another condition on $\chi$, i.e. $\{x \in T | \chi(x) \neq 0\}$ is finite. This is the only kind of semicharacter that we use, and we will refer to them simply as semicharacters.

4. Theorem. If $\chi$ is a semicharacter on $S$, then the map $f \rightarrow \sum_{x \in S} f(x)\chi(x)$ is a multiplicative linear functional on $l(S)$.

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PROOF. Each sum below is finite so we rearrange the terms with impunity.

\[
\left( \sum_{x \in S} f(x) \chi(x) \right) \left( \sum_{y \in S} g(y) \chi(y) \right) = \sum_{x, y \in S} f(x) g(y) \chi(xy) = \sum_{n \in S} \left( \sum_{x \cdot y = n} f(x) g(y) \right) \chi(n) = \sum_{n \in S} f * g(n) \chi(n).
\]

5. Lehmer’s theorem. Let \( N \) be the positive integers under any semigroup operation such that the conditions of \( \S 2 \) are satisfied with \( \varepsilon = 1 \). In addition Lehmer postulates

\[(5.1) \quad xyn \neq n \quad \text{implies} \quad xn \neq n \quad \text{and} \quad yn \neq n.\]

Equation (5.1) is enough to guarantee that \( \chi_n \), the characteristic function of \( A_n \), is a semicharacter on \( N \).

For each \( f \in l(N) \) define \( F \) by

\[ F(n) = \sum_{x \in A_n} f(x) = \sum_{x \in N} f(x) \chi_n(x). \]

Then if \( f * g = h \), we have by Theorem 4 that \( H(n) = F(n)G(n) \) which is Theorem 4 of [2].

6. Tainiter’s theorem. Let \( T \) be a finite, commutative semigroup of idempotents. If \( x \in D_n \), then for some \( y \in T \), \( xy = n \) and \( xyn = n^2 = n \), i.e. \( xy \in A_n \). It follows that \( x \in A_n \) since \( xn = x(xy) = x^2y = xy = n \). So (5.1) holds and Theorem 4 in this case is Theorem 3.1 of [3].

For applications of Theorem 4 to combinatorial analysis see Tainiter [3]. For application to a class of convolution algebras and to elementary number theory see [1].

REFERENCES

1. P. Aizley, Structure theory for a class of convolution algebras, University Microfilms, Order #69-16,475, Ann Arbor, Michigan.

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