COEFFICIENTS OF MEROMORPHIC SCHLICHT FUNCTIONS

PETER L. DUREN

Abstract. This paper presents an elementary proof of a known theorem on the coefficients of meromorphic schlicht functions: if \( f \in \Sigma \) and \( b_k = 0 \) for \( 1 \leq k < n/2 \), then \( |b_n| \leq 2/(n+1) \).

Let \( \Sigma \) denote the class of functions
\[
f(z) = z + b_0 + b_1z^{-1} + b_2z^{-2} + \cdots
\]
analytic and schlicht in \( |z| > 1 \) except for a simple pole at \( \infty \) with residue 1. Let \( \Sigma_0 \) be the subclass of \( \Sigma \) for which \( b_0 = 0 \). It follows from the area theorem that \( |b_1| \leq 1 \), and Schiffer \([7]\) obtained the sharp estimate \( |b_2| \leq 2/3 \). The form of the extremal functions suggested that \( |b_n| \leq 2/(n+1) \) for all \( n \). This has proved to be quite false, but it is true for certain subclasses of \( \Sigma \). For example, the following theorem goes back to Goluzin \([3]\).

Theorem. Let \( f \in \Sigma \) and suppose \( b_1 = b_2 = \cdots = b_{m-1} = 0 \) for some \( m \geq 1 \). Then \( |b_n| \leq 2/(m+1) \), \( n = m, m+1, \ldots, 2m \).

The inequality \( |b_2| \leq 2/3 \) is a special case of this theorem. Jenkins \([5]\) proved the theorem by the method of quadratic differentials. I claimed \([1]\) to give an elementary proof based on the Grunsky inequalities. However, Professor Jenkins has pointed out to me that my proof contains an error, since the square-root transformation \( \sqrt{f(z^2)} \) will introduce a branch point wherever \( f(z^2) = 0 \). Furthermore, Goluzin made essentially the same mistake (see the footnote in \([4, p. 279]\)).

The purpose of this paper is to correct the error in \([1]\) and thus to deduce the theorem in an elementary way from the Grunsky inequalities. The main idea was suggested by Pommerenke’s recent proof \([6]\) that \( |b_2| \leq 2/3 \).

The \( n \)th Faber polynomial \( F_n(w) \) of a function \( f \in \Sigma \) is defined by
\[
F_n[f(z)] = z^n + \sum_{k=1}^{\infty} \beta_{nk}z^{-k}.
\]
The inequality
\[ |\beta_{mn}| \leq 1, \quad n = 1, 2, \ldots, \]
is a very special case of the Grunsky inequalities.

To prove the theorem, it is enough to show \(|b_{2m-1}| \leq 1/m\) and \(|b_{2m}| \leq 2/(2m+1)\). For the first of these inequalities, assume without loss of generality that \(f(0) = 0\) and observe (as in [1]) that
\[ |\beta_{2m-1}| \leq 1/m, \quad \text{by (1)}. \]

To obtain the estimate for \(|b_{2m}|\), assume \(b_0\) is chosen so that \(f(z) \neq 0\) in \(|z| > 1\), and let
\[ g(z) = \sqrt{(f(z^2))} = z \left( 1 + b_0 z^{-2} + b_m z^{-2m-2} + b_{m+1} z^{-2m-4} + \ldots \right)^{1/2}. \]

Let \(F_n^*(w)\) be the \(n\)th Faber polynomial of \(g\), so that
\[ F_n^*[g(z)] = z^n + \sum_{j=1}^{\infty} \beta_{nj} z^{-j}. \]

The proof depends on the following lemma.

**Lemma.** For each fixed \(m\), there exist real numbers \(r_k\) such that
\[ \beta_{2k+1,2m+1} = ((2k+1)/2)b_{m+k} + r_k b_0^m b_{m+1} \quad \text{for } k = 0, 1, \ldots, m. \]

**Proof.** The lemma will be proved by induction on \(k\). For \(k = 0\), it is clear that \(F_1[g(z)] = g(z)\), and one computes
\[ \beta_{1,2m+1} = \frac{1}{2} b_m + \left( \frac{1}{2} \right) b_0^m b_{m+1}, \]
where \(\binom{x}{m} = x(x-1) \cdots (x-n+1)/n!\) is a binomial coefficient. Assuming now that the lemma has been proved for 0, 1, \ldots, \(k-1\), consider
\[ [g(z)]^{2k+1} = z^{2k+1} \left\{ 1 + b_0 z^{-2} + b_m z^{-2m-2} + \ldots \right\}^{k+1/2} \]
\[ = z^{2k+1} \sum_{j=0}^{\infty} \binom{k + \frac{3}{2}}{j} (1 + b_0 z^{-2})^{k-j+1/2} (b_m z^{-2m-2} + \ldots)^j \]
\[ = (1 + b_0 z^{-2})^{k+1/2} z^{2k+1} + (k + \frac{3}{2})(1 + b_0 z^{-2})^{k-1/2} \]
\[ \cdot (b_m + b_{m+1} z^{-2} + \ldots) z^{2(k-m)-1} + O(z^{-2m-2}) \]
\[ = z^{2k+1} + \sum_{j=1}^{k} \binom{k + \frac{3}{2}}{j} b_0^j z^{-j} z^{2(k-j)+1} + \sum_{j=1}^{\infty} \gamma_{kj} z^{-j}, \]
say. A calculation gives

\[ (2) \quad \gamma_{k,2m+1} = \left( \begin{array}{c} k + \frac{1}{2} \\ k + m + 1 \end{array} \right) b_0^{k+m+1} + \left( \begin{array}{c} k + \frac{1}{2} \\ k \end{array} \right) \sum_{n=0}^{k} \left( \begin{array}{c} k - \frac{1}{2} \\ n \end{array} \right) b_0^n b_{m-k-n}. \]

From the form of \([g(z)]^{2k+1}\), it is clear that

\[ F_{2k+1}^*[g(z)] = [g(z)]^{2k+1} - \sum_{j=0}^{k-1} \left( \begin{array}{c} k + \frac{1}{2} \\ k - j \end{array} \right) b_0^{-j} F_{2j+1}^*[g(z)]. \]

Thus by (2) and our inductive hypothesis, we have

\[ \beta_{2k+1,2m+1} = \gamma_{k,2m+1} - \sum_{j=0}^{k-1} \left( \begin{array}{c} k + \frac{1}{2} \\ k - j \end{array} \right) b_0^{-j} [b_{m+j} + r_j b_0^{m+j+1}] \]

\[ = \left( k + \frac{1}{2} \right) b_{m+k} + r_k b_0^{m+k+1}, \]

since

\[ (k + \frac{1}{2}) \left( \begin{array}{c} k - \frac{1}{2} \\ k - j \end{array} \right) = \left( \begin{array}{c} j + \frac{1}{2} \\ k - j \end{array} \right), \quad j = 0, 1, \ldots, k - 1. \]

This proves the lemma.

In view of (1), the lemma gives the inequality

\[ |((2m + 1)/2)b_{2m} + r_m b_0^{2m+1}| \leq 1 \]

for the case in which \(f(z) \neq 0\) in \(|z| > 1\). Now let

\[ f(z) = z + b_m z^{-m} + b_{m+1} z^{-m-1} + \cdots \in \Sigma_0, \]

and let \(E\) be the complement of the range of \(f\). If \(0 \in E\), then (3) gives at once the inequality \(|b_{2m}| \leq 2/(2m+1)\). The proof is complete also if \(r_m = 0\).

Suppose next that \(0 \notin E\) and \(r_m \neq 0\). Assume without loss of generality that \(b_{2m} \geq 0\). If \(\alpha \in E\), then \([f(z) - \alpha]\) has no zeros in \(|z| > 1\), so by (3),

\[ b_{2m} \leq \frac{2}{(2m + 1)} \left[ 1 + r_m \text{Re}\{\alpha^{2m+1}\} \right], \quad \alpha \in E. \]

The proof will be complete, then, if we show that \(\alpha \in E\) can be chosen so that

\[ r_m \text{Re}\{\alpha^{2m+1}\} \leq 0. \]

But if this expression is positive for all \(\alpha \in E\), then since \(E\) is connected and \(0 \notin E\), it follows that \(E\) lies entirely in an open sector with vertex 0 and angle \(\pi/(2m+1)\). On the other hand, 0 is in the convex hull of \(E\),

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
since \( \int_0^\infty f(re^{i\theta})d\theta = 0, r > 1 \). This contradiction shows that \( \alpha \in \mathbb{E} \) can be chosen to satisfy (4), proving that \( b_{2m} \leq 2/(2m+1) \).

Garabedian and Schiffer [2] found that \( \text{Re} \{b_3\} \) is maximized by a function \( f \in \mathcal{S} \) with
\[
f(z) = z + 4i e^{-z} z^{-1} + (\frac{1}{3} + e^{-6}) z^{-3} + \cdots.
\]
However, the above theorem shows that \( |b_3| \leq \frac{1}{3} \) if \( b_1 = 0 \). This can be generalized as follows.

**Theorem.** If \( f \in \mathcal{S} \) and \( |\arg \{b_3\} - \arg \{b_2\}| \leq \pi/2 \), then \( |b_3| \leq \frac{1}{3} \).

**Proof.** Assume that \( b_0 = 0 \) and, after a suitable rotation, that \( b_3 > 0 \) and \( \text{Re} \{b_1\} \geq 0 \). Observe that
\[
|f(z)|^2 = z^2 + 2b_1 + 2b_3 z^{-1} + (2b_3 + b_1) z^{-2} + \cdots.
\]
Hence by (1), \( \text{Re} \{2b_3 + b_1^2\} \leq 1 \), which shows \( b_3 \leq \frac{1}{3} \).

Similar generalizations can be made to higher coefficients.

**References**


University of Michigan, Ann Arbor, Michigan 48104