DECOMPOSITION OF FUNCTION-LATTICES

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Abstract. We give a simple direct proof of the theorem (due to Kaplansky-Blair-Burrill) that the lattice \( C(X, K) \) of all continuous functions defined on the topological space \( X \) with values in the chain \( K \) can be decomposed iff \( X \) contains an open-and-closed subset.

For any topological space \( X \), let \( C(X, K) \) denote the lattice of all \( K \)-valued continuous functions defined on \( X \), where \( K \) is any non-singleton totally ordered set with the order topology. Clearly, if \( A \) is any open-and-closed subspace of \( X \), then \( C(X, K) \) is lattice isomorphic to the direct product \( C(A, K) \times C(X \setminus A, K) \). Improving a technique of Kaplansky [2], Blair and Burrill [1] have shown that a converse holds. We give a simple alternative proof of this result which, in contrast to the proofs of Kaplansky and Blair-Burrill, avoids use of the axiom of choice. For this observation and several other suggestions for improving the presentation we are grateful to the referee.

A sublattice \( L \subseteq C(X, K) \) is adequate provided that, for each \( x \in X \), there are functions \( f, g \in L \) such that \( f(x) \neq g(x) \).

Theorem. If an adequate sublattice \( L \) of \( C(X, K) \) is lattice isomorphic to the direct product \( L_1 \times L_2 \) of lattices \( L_1 \) and \( L_2 \), then there is an open-and-closed subset \( A \subseteq X \) such that \( L_1 \) is lattice isomorphic to \( \{f| A : f \in L_1 \} \) and \( L_2 \) is lattice isomorphic to \( \{f| (X \setminus A) : f \in L_2 \} \).

We first establish a

Lemma. Let \( L_1 \) and \( L_2 \) be lattices and \( K \) be a totally ordered set. If \( \alpha: L_1 \times L_2 \to K \) is a lattice homomorphism, then one of the following holds:

1. For any \( k, k' \in L_2, \alpha(l, k) = \alpha(l, k') \) for any \( l \in L_1 \).
2. For any \( l, l' \in L_1, \alpha(l, k) = \alpha(l', k) \) for any \( k \in L_2 \).

Moreover, if \( \alpha \) is not constant, then precisely one of these holds.

Proof. Note that (1) is equivalent to:

\( 1' \) For any \( k, k' \in L_2, \alpha(l_0, k) = \alpha(l_0, k') \) for some \( l_0 \in L_1 \).

This follows from the observation that

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\[ \alpha(l, k) = \alpha(((l_0, k) \land (l, k \lor k')) \lor (l, k \land k')) \]
\[ = \alpha(((l_0, k') \land (l, k \lor k')) \lor (l, k \land k')) = \alpha(l, k'). \]

Similarly, (2) is equivalent to:

(2') For any \( l, l' \in L_1 \), \( \alpha(l, k_0) = \alpha(l', k_0) \) for some \( k_0 \in L_2 \).

Now, assume that condition (2) fails. Then, there exist \( l, l' \in L_1 \) such that \( \alpha(l, k) \neq \alpha(l', k) \) for any \( k \in L_2 \). To show that condition (1) holds, consider any \( k, k' \in L_2 \). Then,

\[ \alpha(l \land l', k \lor k') \lor \alpha(l \lor l', k \land k') = \alpha(l \lor l', k \lor k'). \]

Since \( K \) is totally ordered and \( \alpha(l \lor l', k \lor k') = \alpha(l \lor l', k \land k') \) implies that \( \alpha(l, k \lor k') = \alpha(l', k \lor k') \) (a contradiction), we conclude that \( \alpha(l \lor l', k \land k') = \alpha(l \lor l', k \lor k') \).

Hence, \( \alpha(l \lor l', k) = \alpha(l \lor l', k') \) so that conditions (1') and (1) hold. Evidently, both conditions hold iff \( \alpha \) is constant.

Proof of the Theorem. Let \( L \) be an adequate sublattice of \( C(X, K) \) and \( \psi: L_1 \times L_2 \to L \) be a lattice isomorphism. For each \( x \in X \), the lattice homomorphism \( \varphi_x: L_1 \to K \), defined by \( \varphi_x(f) = f(x) \), is not constant. From the preceding lemma \( \varphi_x \circ \psi: L_1 \times L_2 \to K \) satisfies one, and only one, of the conditions (1) and (2). Define \( A = \{ x \in X: \varphi_x \circ \psi \) satisfies condition (1) \}. It follows easily that \( A \) and \( X \setminus A \) are disjoint closed sets. Finally, define \( \theta: L_1 \to \{ f \in L_1: f \in L \} \) by \( \theta(l) = f \) \( A \), where \( f = \psi(l, k_0) \) for some \( k_0 \in L_2 \). It follows directly that \( \theta \) is a lattice isomorphism. Similarly, one considers \( X \setminus A \) so that the proof of the theorem is complete.

Remarks. An easy corollary is that a topological space \( X \) is connected iff, for any totally ordered set \( K \), there is no adequate sublattice \( L \subseteq C(X, K) \) which is lattice isomorphic to the direct product \( L_1 \times L_2 \) of two lattices \( L_1 \) and \( L_2 \), neither of which is a singleton. Hence, a topological space \( X \) is connected iff every extension of \( X \) is connected (where an extension of \( X \) is any topological space that contains \( X \) as a dense subspace).

References


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