EXTENSIONS OF DERIVATIONS

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Abstract. We show that for a class of algebras including separable algebras one can extend derivations of the center to derivations of the algebra.

The following theorem was proved in the special cases that $C$ is a field by Hochschild [Ho] and $C$ is a semilocal ring by Roy and Sridharan [R, S] [and for any $C$ in [Kn]]. It is also a trivial consequence of a more general result proved by a short cohomological argument.

**Theorem 1.** Let $A$ be an algebra separable over its center $C$ and $M$ be an $A \otimes_C A^{op}$-module. Then any derivation $d : C \rightarrow M^A$ extends to a derivation $\bar{d} : A \rightarrow M$.

Since an algebra separable over its center $C$ is $C$-projective [A, G, p. 379], Theorem 1 follows from

**Theorem 2.** Let $A$ be a $C$-algebra, projective over $C$, of Hochschild dimension one and let $M$ be an $A \otimes_C A^{op}$-module. Then any derivation $d : C \rightarrow M^A$ extends to a derivation $\bar{d} : A \rightarrow M$.

Proof. Let $B$ be the split extension of $A$ by $M$. That is, $B$ is the additive group $A \oplus M$ with $(a, m)(a', m') = (aa', am' + ma')$. If we let $C$ operate on $B$ by $c(a, m) = (ca, cm + dc \cdot a) = (a, m)c$, then $B$ is a $C$-algebra and the projection of $B$ to $A$ is a $C$-algebra homomorphism. It is also $C$-linearly split since $A$ is $C$-projective. Thus the extension is an element of $H^3_C(A, M)$ which is zero by hypothesis. This means that there is a $C$-algebra splitting of $B \rightarrow A$, the second coordinate of which is easily seen to be a derivation extending $d$.

**Corollary 1.** Let $A$, $C$, $M$ be as above and $A_0$ be a separable $C$-subalgebra of $A$. Then any derivation $d : A_0 \rightarrow M$ which takes $C$ to $M^A$ can be extended to a derivation $\bar{d} : A \rightarrow M$.

Proof. First restrict to $C$, then extend to $A$. The difference, on $A_0$, is $C$-linear and hence inner.

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Corollary 2. Let $A$ be a $R$-algebra separable over its center $C$ and let $M$ be a $A \otimes_C A^{op}$-module. Then $H_k^1(A, M) \cong \text{Der}_R(C, M^A)$.

Proof. It is evident that any derivation of $A$ to $M$ restricts to a derivation $C \to M^A$.

References


