ON DIFFERENTIABILITY OF MINIMAL SURFACES AT A BOUNDARY POINT

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Abstract. Let \( F(z) = \{u(z), v(z), w(z)\}, |z| < 1 \), represent a minimal surface spanning the curve \( \Gamma: \{U(s), V(s), W(s)\}, s \) being the arc length. Suppose \( \Gamma \) has a tangent at a point \( P \). Then \( F(z) \) is differentiable at this point if \( U'(s), V'(s), W'(s) \) satisfy a Dini condition at \( P \).

Let \( \Gamma \) be a closed rectifiable Jordan curve in Euclidean 3-space, and let \( F(z) = \{u(z), v(z), w(z)\} \), defined in the disk \( \{z: |z| < 1\} \) (\( z = x + iy = re^{i\theta} \)), represent a generalized minimal surface spanning \( \Gamma \), i.e.

1. \( u(z), v(z), w(z) \) are harmonic in \( |z| < 1 \) and continuous in \( |z| \leq 1 \);
2. \( x, y \) are isothermal parameters in \( |z| \leq 1 \), i.e.

\[
\begin{align*}
1. & \quad |F_x|^2 = u_x^2 + v_x^2 + w_x^2 = |F_y|^2 = u_y^2 + v_y^2 + w_y^2, \\
2. & \quad F_x \cdot F_y = u_xu_y + v_xv_y + w_xw_y = 0;
\end{align*}
\]

3. \( F(e^{i\theta}), 0 \leq \theta < 2\pi, \) is a homeomorphism of \( |z| = 1 \) with \( \Gamma \).

The components \( u, v, w \) of the vector \( F \) are the real parts of analytic functions in \( |z| < 1 \):

\[
\lambda(z) = u(z) + iu^*(z), \quad \mu(z) = v(z) + iv^*(z), \quad \nu(z) = w(z) + iw^*(z).
\]

Recently various theorems dealing with the boundary behavior of conformal maps in the plane have been extended to minimal surfaces by J. C. C. Nitsche [2], D. Kinderlehrer [1], S. E. Warschawski [3], and other authors. Nitsche’s paper contains a survey of prior work on the boundary behavior of minimal surfaces. The purpose of this note is to present a local result concerning differentiability of minimal surfaces at a given point on the boundary. In fact, our result extends a theorem of Warschawski on conformal mapping in the plane, namely Theorem 1 in [4].

Theorem. Suppose \( \{U(s), V(s), W(s)\} \) denotes the parametric representation of \( \Gamma \) in terms of arc length. Assume \( P_0 = \{U(s_0), V(s_0), W(s_0)\} \) is a point of \( \Gamma \) and that \( \Gamma \) has a tangent at \( P_0 \), i.e. \( U'(s_0), V'(s_0), W'(s_0) \) exist.2

Received by the editors May 8, 1970.

AMS 1969 subject classifications. Primary 5304, 3040, 3062.

Key words and phrases. Complex analysis, minimal surfaces, boundary behavior.

1 Research supported in part by U. S. Air Force Grant AFOSR-68-1514.

2 We assume \( \{U'(s_0), V'(s_0), W'(s_0)\} \) represents the unit tangent to \( \Gamma \) at \( P_0 \).

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Suppose that there exists a nondecreasing, continuous function \( \omega(t) \geq 0, 0 \leq t \leq a (a > 0) \), such that

\[
\int_0^a \frac{\omega(t)}{t} \, dt < \infty
\]

and

\[
| U'(s) - U'(s_0) | \leq \omega\left( | s - s_0 | \right),
\]

\[
| V'(s) - V'(s_0) | \leq \omega\left( | s - s_0 | \right),
\]

\[
| W'(s) - W'(s_0) | \leq \omega\left( | s - s_0 | \right),
\]

for all points \( \{ U(s), V(s), W(s) \} \) in a neighborhood of \( P_0 \) at which \( U'(s), V'(s), W'(s) \) exist.\(^3\)

Let \( F(e^{\theta_0}) = P_0 \). Then

\[
\lim_{z \to z_0} \frac{\lambda(z) - \lambda(z_0)}{z - z_0} = \lambda'(z_0) \quad (z_0 = e^{\theta_0})
\]

exists for unrestricted approach in \( | z | \leq 1 (z \neq z_0) \), and

\[
\lim_{z \to z_0} \lambda'(z) = \lambda'(z_0)
\]

for \( z \) in any Stolz angle with vertex at \( z_0 \). The same holds for \( \mu(z) \) and \( \nu(z) \).\(^4\)

**Proof.** Without loss of generality we may assume \( U'(s_0) = 1, V'(s_0) = 0, W'(s_0) = 0 \). Under the conditions of the theorem Warshawski proved the following facts (see [3, Part II, §§2–7]):

There is an interval \( [\theta_1, \theta_2] \) containing \( \theta_0 \) in its interior, a constant \( a > 1 \), and a sector \( S = \{ z = re^{i\theta} : 0 < r < 1, \theta_1 < \theta < \theta_2 \} \) such that, if \( \varphi(\xi) \) maps \( | \xi | < 1 \) conformally onto \( S (\varphi(1) = e^{\theta_0}) \) and

\[
f = \log \left( \frac{X + a}{X} \right) = \log \left[ \frac{X + a}{X} \right],
\]

then \( \lim_{r \to 1} \text{Im} \, f(\xi) \) exists for unrestricted approach in \( | \xi | \leq 1 \) as well as \( \lim_{r \to 1} \varphi(\xi) = \varphi(1) \). The same holds for \( i\bar{g} = i[\mu_\theta/(\lambda_\theta + \alpha)] \circ \varphi = i[\lambda_\theta/(\lambda_\theta + \alpha)] \) and \( i\bar{h} = i[\nu_\theta/(\lambda_\theta + \alpha)] \circ \varphi = i[\lambda_\theta/(\lambda_\theta + \alpha)] \).\(^5\) Let \( \Phi(\xi) \)

\[^3\] It should be noted that under the hypotheses of the theorem one can show the existence of a subarc \( \gamma \) containing \( P_0 \) in its interior and having the following property: \( \Delta s \leq c(P_1P_2) \) where \( c \) is a constant, \( c > 1 \), \( \Delta s \) is the length of the subarc of \( \gamma \) between \( P_1, P_2, P \), and \( (P_1P_2) \) is the chordal distance.

\[^4\] The author wishes to express his indebtedness to the referee for his remark simplifying the statement of the theorem.

\[^5\] Isothermal relations (1) and (2) are essential in obtaining these results.
Let $\Phi(\zeta) + i\Phi_2(\zeta)$ ($\Phi = \text{Re } \Phi$, $\Phi_2 = \text{Im } \Phi$) be holomorphic in $|\zeta| < 1$. Assume

\[ \lim_{\rho \to 1} \Phi(\rho) = \Phi(1) = \Phi_1(1) + i\Phi_2(1) \]

exists, and

\[ \lim_{\zeta \to 1} \Phi_2(\zeta) = \Phi_2(1) \]

for unrestricted approach in $|\zeta| < 1$. Then by a theorem of Warschawski [5, p. 315, Theorem II] one has

\[ \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta \left\{ \exp[\Phi(e^{it}) - \Phi(1)] - 1 \right\} dt = 0. \]

Also, it is readily seen from the proof of this theorem that

There exists a subarc $\gamma$ of $|\zeta| = 1$ with midpoint $\zeta = 1$

(iii) such that $\lim_{\rho \to 1} \Phi(\rho e^{it}) = \Phi(e^{it})$ exists for almost all $e^{it} \in \gamma$, $\Phi(e^{it})$ is integrable along $\gamma$, and

\[ \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta |\Phi(e^{it}) - \Phi(1)|^2 dt = 0. \]

Since $\varphi'(\zeta) \neq 0$, we can define $\log \varphi'(\zeta)$ as a single valued analytic function in $|\zeta| < 1$. By our remarks at the beginning, $\Phi_0(\zeta) = \overline{f}(\zeta) + \log \varphi'(\zeta)$ satisfies (3) and (4) and we can apply (5) to $\Phi_0(\zeta)$ to obtain

\[ \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta \left\{ \exp[\log(\lambda_\theta(e^{it}) + \alpha) + \varphi'(e^{it}) - \log(\lambda_\theta(1) + \alpha) - \log \varphi'(1)] \right\} dt = 1 \]

which implies

\[ \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta \frac{(\lambda_\theta(e^{it}) + \alpha)\varphi'(e^{it})}{(\lambda_\theta(1) + \alpha)\varphi'(1)} e^{it} dt = 1. \]

Letting $\varphi(e^{it}) = e^{it}$ and changing the variable of integration ($\varphi(e^{it}) = e^{it}$) we readily obtain
\[
\lim_{\xi \to \theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} (\lambda(e^{i\theta}) + \alpha)e^{i\theta} \, d\theta = (\lambda(e^{i\theta_0}) + \alpha)e^{i\theta_0}.
\]

Now,
\[
e^{i\theta_0} \left[ \frac{\lambda(e^{i\xi}) - \lambda(e^{i\theta_0})}{\xi - \theta_0} \right]
\]
\[
= \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} \lambda(e^{i\theta})(e^{i\theta_0} - e^{i\theta}) \, d\theta + \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} \lambda(e^{i\theta})e^{i\theta} \, d\theta,
\]

since \(\lambda(e^{i\theta})\) is absolutely continuous [3]. By (8), the second term in (9) approaches the limit \(\lambda(e^{i\theta_0})e^{i\theta_0}\), and the first term approaches 0 as \(\xi \to \theta_0\), since
\[
\frac{1}{|\xi - \theta_0|} \int_{\theta_0}^{\xi} |\lambda(e^{i\theta})| \cdot |e^{i\theta_0} - e^{i\theta}| \, d\theta \leq \frac{|e^{i\xi} - e^{i\theta_0}|}{\xi - \theta_0} \int_{\theta_0}^{\xi} |\lambda(e^{i\theta})| \, d\theta
\]
and \(\lambda(e^{i\theta})\) is integrable. Therefore,
\[
\lim_{\xi \to \theta_0} \frac{\lambda(e^{i\xi}) - \lambda(e^{i\theta_0})}{\xi - \theta_0} = \lambda(e^{i\theta_0}).
\]

From (10) it follows that
\[
\lim_{e^{i\theta_0} \to e^{i\theta_0}} \frac{\lambda(e^{i\theta}) - \lambda(e^{i\theta_0})}{e^{i\theta} - e^{i\theta_0}} = \lambda'(e^{i\theta_0})
\]
exists.

The function \((\lambda(z) - \lambda(e^{i\theta_0}))/(z - e^{i\theta_0})\) is holomorphic in \(|z| < 1\) and by (11) and by the fact that \(\lambda(z)\) is continuous on \(|z| = 1\) it is bounded on \(|z| = 1\). The continuity of \(\lambda(z)\) in \(|z| \leq 1\) also ensures that
\[
\frac{\lambda(z) - \lambda(e^{i\theta_0})}{z - e^{i\theta_0}} = O\left(\frac{1}{|z - e^{i\theta_0}|}\right) \quad \text{for} \quad |z| < 1.
\]

Therefore, by a theorem of Phragmén-Lindelöf
\[
((\lambda(z) - \lambda(e^{i\theta_0}))/(z - e^{i\theta_0})
\]
is bounded in \(|z| < 1\). Hence, by a theorem of Lindelöf,
\[
\lim_{z \to e^{i\theta_0}} \frac{\lambda(z) - \lambda(e^{i\theta_0})}{z - e^{i\theta_0}} = \lambda'(e^{i\theta_0})
\]
for unrestricted approach in \(|z| \leq 1\). The second equation,
\[
\lim_{z \to z_0} \lambda'(z) = \lambda'(z_0) \text{ in any Stolz angle with vertex at } z_0, \text{ is a well-known consequence of the first.}
\]

We can apply (7) to \( i\hat{g}(\xi) \) and obtain
\[
(12) \quad \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta \left| \frac{\bar{\mu}_\theta(e^{i t})}{\bar{\lambda}_\theta(e^{i t}) + \alpha} - \frac{\bar{\mu}_\theta(1)}{\bar{\lambda}_\theta(1) + \alpha} \right|^2 dt = 0.
\]

Also, we can apply (6) to \( 2\hat{f}(\xi) \) and conclude that
\[
(13) \quad \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta \left\{ \left| \frac{\bar{\lambda}_\theta(e^{i t}) + \alpha}{\bar{\lambda}_\theta(1) + \alpha} \right|^2 - 1 \right\} dt = 0.
\]

Thus,
\[
(14) \quad \frac{1}{|\eta|} \int_0^\eta \left| \frac{\bar{\lambda}_\theta(e^{i t}) + \alpha}{\bar{\lambda}_\theta(1) + \alpha} \right|^2 dt \leq M_0
\]

for \(|\eta| \leq \eta_0\) and some constant \(M_0\).

By Schwarz's inequality,
\[
\frac{1}{\eta} \int_0^\eta \left| \frac{\bar{\mu}_\theta(e^{i t})}{\bar{\lambda}_\theta(e^{i t}) + \alpha} - \frac{\bar{\mu}_\theta(1)}{\bar{\lambda}_\theta(1) + \alpha} \right| dt \leq \left( \frac{1}{\eta} \int_0^\eta \left| \frac{\bar{\mu}_\theta(e^{i t})}{\bar{\lambda}_\theta(e^{i t}) + \alpha} - \frac{\bar{\mu}_\theta(1)}{\bar{\lambda}_\theta(1) + \alpha} \right|^2 dt \right)^{1/2}
\]

\[
\cdot \left( \left| \frac{\bar{\lambda}_\theta(1) + \alpha}{\bar{\lambda}_\theta(1) + \alpha} \right|^2 \frac{1}{\eta} \int_0^\eta \left| \bar{\lambda}_\theta(e^{i t}) + \alpha \right|^2 dt \right)^{1/2}.
\]

(12), (14) and (15) imply
\[
(16) \quad \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta \left| \frac{\bar{\mu}_\theta(e^{i t})}{\bar{\lambda}_\theta(e^{i t}) + \alpha} - \frac{\bar{\mu}_\theta(1)}{\bar{\lambda}_\theta(1) + \alpha} \right| dt = 0.
\]

Since \(\varphi'(e^{i \xi})\) is bounded in a neighborhood of \(\xi = 1\), we also have
\[
(17) \quad \lim_{\eta \to 0} \frac{1}{\eta} \int_0^\eta \left| \frac{\bar{\mu}_\theta(e^{i t})}{\bar{\lambda}_\theta(e^{i t}) + \alpha} - \frac{\bar{\mu}_\theta(1)}{\bar{\lambda}_\theta(1) + \alpha} \right| |\varphi'(e^{i t})| dt = 0.
\]

Changing the variable of integration, as in the case of \(\lambda_\theta(e^{i \theta})\), one concludes from (17) that
\[
(18) \quad \lim_{\xi \to 0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^\xi \left| \mu_\theta(e^{i \theta}) (\lambda_\theta(e^{i \theta}) + \alpha) - \mu_\theta(e^{i \theta_0}) (\lambda_\theta(e^{i \theta}) + \alpha) \right| d\theta = 0.
\]

Thus,
\[(\lambda_{\theta}(e^{\theta_0}) + \alpha) \lim_{\xi \to \theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} \mu_{\theta}(e^{\theta}) \, d\theta \]

\[= \mu_{\theta}(e^{\theta_0}) \lim_{\xi \to \theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} (\lambda_{\theta}(e^{\theta}) + \alpha) \, d\theta. \]

By (10)

\[\lim_{\xi \to \theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} (\lambda_{\theta}(e^{\theta}) + \alpha) \, d\theta = \lambda_{\theta}(e^{\theta_0}) + \alpha, \]

and \(\lambda_{\theta}(e^{\theta_0}) + \alpha \neq 0.\)

Therefore (19) and (20) imply

\[\lim_{\xi \to \theta_0} \frac{1}{\xi - \theta_0} \int_{\theta_0}^{\xi} \mu_{\theta}(e^{\theta}) \, d\theta = \mu_{\theta}(e^{\theta_0}). \]

From (21) one infers that

\[\lim_{z \to e^{\theta_0}} \frac{\mu(z) - \mu(e^{\theta_0})}{z - e^{\theta_0}} \]

exists for unrestricted approach in \(|z| \leq 1\), exactly the same way as we showed this limit exists in the case of \(\lambda(z)\). We deal with \(\nu(z)\) in a similar fashion.

It should be noted that one may assume only the subarc \(\gamma\) to be rectifiable and obtain the same result with slight modification of our proof.

**Bibliography**


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