PARACOMPACTNESS AND ELASTIC SPACES

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Abstract. This paper gives a characterization of paracompactness, and introduces the notion of an elastic space which generalizes the concept of a stratifiable (in particular, metric) space.

1. Introduction. In this note we shall give a characterization of paracompactness which is formally weaker than our previous characterizations [4, Theorem 2], [5, Theorem 3], [6, Theorem 1] concerning linearly cushioned refinements. Furthermore, we shall define a new generalization of metric spaces and stratifiable spaces, called "elastic spaces," by introducing the notion of an "elastic base."

Definition 1. Let \( \mathcal{U} \) be a collection of subsets of a set \( X \), and let \( \mathcal{A} \) be a relation on \( \mathcal{U} \) (i.e., \( \mathcal{A} \subseteq \mathcal{U} \times \mathcal{U} \)). We shall often write \( U \mathcal{A} V \) instead of \( (U, V) \in \mathcal{A} \). The relation \( \mathcal{A} \) is said to be a framed relation on \( \mathcal{U} \) (or a framing of \( \mathcal{U} \)) provided for every \( U, V \in \mathcal{U} \), if \( U \cap V \neq \emptyset \), then \( U \mathcal{A} V \) or \( V \mathcal{A} U \). We say \( \mathcal{A} \) is a well-framed relation on \( \mathcal{U} \) provided \( \mathcal{A} \) is a framing of \( \mathcal{U} \) and for every \( x \in X \), there exists an \( \mathcal{A} \)-smallest \( U_x \subseteq U \) containing \( x \) (i.e., if \( x \in U, U \in \mathcal{U} \), and \( U \neq U_x \), then \( (U, U_x) \notin \mathcal{A} \)).

Definition 2. A collection \( \mathcal{U} \) is said to be framed in a collection \( \mathcal{V} \) with frame map \( f: \mathcal{U} \to \mathcal{V} \) provided there exists a framed relation \( \mathcal{A} \) on \( \mathcal{U} \) such that for every subcollection \( \mathcal{U}' \subseteq \mathcal{U} \) which has an \( \mathcal{A} \)-upper bound (i.e., there exists \( U \in \mathcal{U} \) so that \( U \mathcal{A} U' \) for every \( U' \in \mathcal{U}' \)) we have \( \text{cl}(\mathcal{U}' \cup U) \subseteq \cup f(\mathcal{U}') \). If in addition \( \mathcal{A} \) is a well-framed relation on \( \mathcal{U} \), we say that \( \mathcal{U} \) is well-framed in \( \mathcal{V} \). Finally, if \( \mathcal{U} \) is framed in \( \mathcal{V} \) and \( \mathcal{A} \) is also a transitive relation, then \( \mathcal{U} \) is called elastic in \( \mathcal{V} \), or an elastic refinement of \( \mathcal{V} \) when \( \mathcal{U} \) and \( \mathcal{V} \) are covers of \( X \).

Theorem 1. Let \( X \) be a regular space. A necessary and sufficient condition that \( X \) be paracompact is that every open cover of \( X \) have an open elastic refinement.

2. Proof of Theorem 1. The proof follows from the next two lemmas.

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1 This note is a revision of a manuscript of H. Tamano which was found by J. Nagata after the death of Professor Tamano.
Lemma 1. Let $X$ be a regular space. A necessary and sufficient condition for $X$ to be paracompact is for every open cover of $X$ to have an open refinement which is well-framed in it.

Proof. We shall prove the sufficiency. The proof is similar to that of Theorem 1 in [6]. Let $\mathcal{U}$ be an open cover of $X$. Let $\mathcal{U}$ be an open refinement of $\mathcal{U}$ which is well-framed in $\mathcal{U}$ with respect to the well-framed relation $\mathcal{R}$ on $\mathcal{U}$ and frame map $f : \mathcal{U} \to \mathcal{V}$. Let $H_U = U - \bigcup \{ U' \in \mathcal{U} : U' \cap U \text{ and } U' \neq U \}$, and $\mathcal{K} = \{ H_U : U \in \mathcal{U} \}$. We now show that $\mathcal{K}$ is a cushioned refinement of $\mathcal{U}$ with cushion map $g : \mathcal{K} \to \mathcal{V}$ defined by $g(H_U) = f(U)$ (these terms are defined in [6]) and conclude that $X$ is paracompact by [3, Theorem 1.1, p. 309]. It is easy to see that $\mathcal{K}$ is a cover of $X$ since $\mathcal{R}$ is well-framed. It remains to show that $\mathcal{K}$ is cushioned in $\mathcal{U}$. Let $\mathcal{K}' \subseteq \mathcal{K}$, and suppose $x \in \bigcup g(\mathcal{K}')$. Let $U_x$ be an $\mathcal{R}$-smallest element of $\mathcal{U}$ containing $x$. Clearly $U_x$ is an open neighborhood of $x$ missing $H_{U_x}$ for all $U \neq U_x$ such that $U \cap U_x \neq \emptyset$. Further, $\mathcal{U}' = \{ U \in \mathcal{U} : U \cap U_x \neq \emptyset \}$ and $H_{U_x} \in \mathcal{K}'$ has an $\mathcal{R}$-upper bound. Hence $cl(\bigcup \mathcal{U}') \subseteq \bigcup f(\mathcal{U}') \subseteq \bigcup g(\mathcal{K}')$ because $\mathcal{U}$ is framed in $\mathcal{V}$. Therefore, there exists an open neighborhood $N$ of $x$ missing $cl(\bigcup \mathcal{U}')$. Finally, if $U$ is not $\mathcal{R}$-related to $U_x$, then $U \cap U_x = \emptyset$ since $\mathcal{R}$ is a framing of $\mathcal{U}$. Thus, $U_x \cap N$ is an open neighborhood of $x$ missing $U \cap U_x'$, and we have $cl(\bigcup \mathcal{K}') \subseteq \bigcup g(\mathcal{K}')$. This completes the proof.

Lemma 2. Let $\mathcal{U}$ be a cover of a set $X$, and let $\preceq$ be a transitive relation which is a framing of $\mathcal{U}$. Then there exists a well-framed relation $\mathcal{R}$ on $\mathcal{U}$ so that every subset of $\mathcal{U}$ with an $\mathcal{R}$-upper bound has a $\preceq$-upper bound.

Proof. Let $\preceq^*$ be a well-order on an index set for $\mathcal{U}$ such that $\mathcal{U} = \{ U_0, U_1, \cdots, U_\alpha, \cdots : \alpha < * \eta \}$. Let $\Lambda [\alpha] = \{ U \in \mathcal{U} : \alpha \preceq U \}$ for all $\alpha < * \eta$. Well-order $\Lambda [0]$ in any manner and denote the well-order by $\preceq \leq_0$. Suppose a reflexive and antisymmetric relation $\leq \leq_\beta$ has been defined on $\bigcup \{ \Lambda [\gamma] : \gamma \leq * \beta \}$ for all $\beta < * \alpha$ in such a way that $\leq \leq_\beta$ is an extension of $\leq \leq_\delta$ whenever $\delta \leq * \beta$. Put $\Lambda' [\alpha] = \Lambda [\alpha] - \bigcup \{ \Lambda [\beta] : \beta < * \alpha \}$; well-order $\Lambda' [\alpha]$ and denote the order by $\preceq \leq_\alpha$. Also define $U \leq_\alpha U'$ if $U' \subseteq \Lambda' [\alpha]$ and $U \subseteq \Lambda [\alpha] \cap (\bigcup \{ \Lambda [\beta] : \beta < * \alpha \})$. Let $\preceq \leq_\alpha$ be the relation on $\bigcup \{ \Lambda [\beta] : \beta < * \alpha \}$ generated by $\preceq \leq_\beta : \beta < * \alpha$ and $\preceq \leq_\delta$. Finally, let $\mathcal{R}$ be the reflexive and antisymmetric relation generated by $\{ \preceq \leq_\alpha : \alpha < * \eta \}$. First we shall show that $\mathcal{R}$ is a framing of $\mathcal{U}$. If $U, U' \in \mathcal{U}$ and $U \cap U' \neq \emptyset$, then either $U \preceq U'$ or $U' \preceq U$ since $\preceq$ is a framing of $\mathcal{U}$.

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2 The authors would like to thank E. Michael for some helpful suggestions concerning this result.
Suppose that \( \mathcal{U} \subseteq \mathcal{U}' \). Let \( \alpha_0 \) be the first index such that \( \mathcal{U}' \subseteq \Delta [\alpha_0] \). Since \( \leq \) is transitive, we have \( \mathcal{U} \subseteq \Delta [\alpha_0] \), and thus by the definition of \( \mathcal{U} \) we know \( \mathcal{U}' \cap \mathcal{U} \) or \( \mathcal{U}' \cap \mathcal{U} \).

Next we show that if \( \mathcal{U} \) has an \( \mathcal{U} \)-upper bound, then \( \mathcal{U} \) has a \( \leq \)-upper bound. To do this we first note that if \( \mathcal{U} \cap \mathcal{U}' \), then for the first index \( \alpha \) such that \( \mathcal{U} \subseteq \mathcal{U}' \subseteq \{ \Delta [\beta] : \beta \leq \alpha \} \) we have \( \mathcal{U} \subseteq \mathcal{U}' \subseteq \{ \Delta [\beta] : \beta \leq \alpha \} \). To see this, let \( \gamma \) be the first index such that \( \mathcal{U} \subseteq \gamma \). Since \( \leq \) is a relation on \( \{ \Delta [\beta] : \beta \leq \gamma \} \) we know \( \alpha \leq \gamma \). If \( \alpha < \gamma \), then \( \mathcal{U} \subseteq \mathcal{U}' \subseteq \Delta [\gamma] \) so \( \mathcal{U} \subseteq \mathcal{U}' \) are not related by \( \leq \gamma \).

By the definition of \( \leq \), we must have \( \mathcal{U} \subseteq \mathcal{U}' \) for some \( \beta < \gamma \), but this contradicts the definition of \( \gamma \). Thus, \( \alpha = \gamma \), and \( \mathcal{U} \subseteq \mathcal{U}' \).

Further, not both of \( \mathcal{U} \) and \( \mathcal{U}' \) are in \( \{ \Delta [\beta] : \beta < \delta \} \) for any \( \delta < \alpha \), so \( \mathcal{U} \) and \( \mathcal{U}' \) are not related by \( \leq \delta \). Thus, \( \mathcal{U} \subseteq \mathcal{U}' \), from which it follows that \( \mathcal{U}' \subseteq \Delta [\alpha] \) and \( \mathcal{U} \subseteq \Delta [\alpha] \). Now suppose \( \mathcal{U} \) is a subcollection of \( \mathcal{U} \) which has an \( \mathcal{U} \)-upper bound. Let \( \mathcal{U}' \) be an \( \mathcal{U} \)-upper bound of \( \mathcal{U} \). Let \( \alpha_0 \) be the first index such that \( \mathcal{U}' \subseteq \Delta [\alpha_0] \). We now show that \( \mathcal{U} \) has \( \mathcal{U}_\alpha \) for \( \leq \)-upper bound. Let \( \mathcal{U} \subseteq \mathcal{U} \) and let \( \alpha_1 \) be the first index such that \( \mathcal{U} \subseteq \Delta [\alpha_1] \). If \( \alpha_1 \leq \alpha_0 \), then \( \alpha_0 \) is the first index such that \( \mathcal{U} \subseteq \Delta [\alpha_0] \). Hence \( \mathcal{U} \cap \mathcal{U}' \) implies \( \mathcal{U} \subseteq \mathcal{U}' \) as noted above. In particular \( \mathcal{U} \subseteq \mathcal{U}_\alpha \) by definition of \( \Delta [\alpha_0] \). If \( \alpha_0 < \alpha_1 \) then \( \alpha_1 \) is the first index such that \( \mathcal{U} \subseteq \Delta [\alpha_1] \). Hence \( \mathcal{U} \cap \mathcal{U}' \) implies \( \mathcal{U} \subseteq \Delta [\alpha_1] \), but this contradicts the fact that \( \mathcal{U} \subseteq \Delta [\alpha_0] \).

Finally we show that every nonempty subset \( \mathcal{U}' \) of \( \mathcal{U} \) has an \( \mathcal{U} \)-smallest element. Let \( \alpha \) be the first index such that \( \mathcal{U}' \cap \Delta [\alpha] \neq \emptyset \). Then \( \mathcal{U}' \cap \Delta [\alpha] \subseteq \Delta [\alpha] \). Since \( \{ \Delta [\alpha] : \leq \alpha \} \) is a well-ordered subset of \( \mathcal{U} \), there exists a \( \leq \)-first element of \( \mathcal{U}' \cap \Delta [\alpha] \) which is an \( \mathcal{U} \)-smallest element of \( \mathcal{U}' \). Thus \( \mathcal{U} \) is well-framed, and this completes the proof.

**Proof of Theorem 1.** We need only prove the sufficiency. Let \( \mathcal{U} \) be an open cover of \( X \), and let \( \mathcal{W} \) be an open elastic refinement of \( \mathcal{U} \). By Lemma 2, it is easy to see that there is a well-framed relation on \( \mathcal{W} \), and that \( \mathcal{W} \) is well-framed in \( \mathcal{U} \). Hence \( X \) is paracompact by Lemma 1.

3. **Elastic spaces.** According to J. G. Ceder [2], a collection \( P \) of ordered pairs \( P = (P_1, P_2) \) of subsets of a space \( X \) is called a **pair base** for \( X \) provided that \( P_1 \) is open for all \( P \in P \) and that for every \( x \in X \) and for every open set \( U \) containing \( x \), there exists a \( P \in P \) such that \( x \in P_1 \subseteq P_2 \subseteq U \). Further, he called a \( T_1 \)-space an \( M_3 \)-space (renamed **stratifiable space** by C. J. R. Borges [1]) provided it has a \( \sigma \)-cushioned pair base \( P \). A pair base \( P \) is said to be **\( \sigma \)-cushioned** provided \( P \).

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For every $n$ and every $P_n' \subseteq P_n$ we have $\text{cl}(\bigcup \{ P_1 : P \in P_n' \}) \subseteq \bigcup \{ P_2 : P \in P_n' \}$.

**Definition 3.** A pair base $P$ for a space $X$ is said to be an elastic base if there is a framing of $P_1 = \{ P_1 : P = (P_1, P_2) \in P \}$ such that $P_1$ is elastic in $P_2 = \{ P_2 : P = (P_1, P_2) \in P \}$ with respect to the map $f(P_1) = P_2$. A $T_1$-space with an elastic base is called an elastic space.

**Theorem 2.** Every subspace of an elastic space is an elastic space. Every metrizable space, and more generally every stratifiable space, is an elastic space. Every elastic space is paracompact.

**Proof.** The first statement is obvious. Let $X$ be a stratifiable space with a $\sigma$-cushioned pair base $P = \bigcup_{n=1}^{\infty} P_n$. We may assume that $\{ P_n : n = 1, 2, \ldots \}$ is a partition of $P$. Let $\leq_n$ be a well-order on $P_n$ for each $n$, and define a well-order $\leq$ on $P$ as follows: For $P, P' \in P$ we say $P \leq P'$ if and only if either (1) $P, P'$ are in the same $P_n$ and $P_n \leq P'_n$, or (2) $P \in P_n, P' \in P_m$, and $n < m$. Then $P$ obviously is an elastic base. Since an elastic space is regular, it follows from Theorem 1 that every elastic space is paracompact.

**Example.** (An elastic space which is not a stratifiable space.) Let $X = [0, \Omega]$ be the set of ordinals less than or equal to the first uncountable ordinal. Let the topology on $X$ be the weakest topology stronger than the order topology for which every point is isolated except $\Omega$. Construct an elastic base for $X$ as follows. Let $U_\alpha = (\alpha, \Omega]$ for all $\alpha < \Omega$, and let $P' = \{ (U_\alpha, U_\alpha) : \alpha \in [0, \Omega) \}$ and order $P'$ by the usual order on the index set $[0, \Omega)$. Let $W_\alpha = \{ \alpha \}$ for all $\alpha < \Omega$, and let $P'' = \{ (W_\alpha, W_\alpha) : \alpha \in [0, \Omega) \}$ and order $P''$ by the usual order on the index set $[0, \Omega)$. Finally, set $P = P' \cup P''$ and order $P$ so that every element of $P'$ precedes every element of $P''$. Then $P$ is an elastic base for $X$, so $X$ is an elastic space. Clearly, $X$ is not stratifiable because the closed set $\{ \Omega \}$ is not a $G_\delta$ in $X$ (see [2, Theorem 2.2, p. 106]).

**Conjecture.** Every closed continuous image of an elastic space is an elastic space.

**References**


*Tamaño called this statement a theorem in his manuscript, but he did not give a complete proof; so it is stated here as a conjecture.*

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