PARACOMPACTNESS AND ELASTIC SPACES

HISAHIRO TAMANO¹ AND J. E. VAUGHAN

Abstract. This paper gives a characterization of paracompactness, and introduces the notion of an elastic space which generalizes the concept of a stratifiable (in particular, metric) space.

1. Introduction. In this note we shall give a characterization of paracompactness which is formally weaker than our previous characterizations [4, Theorem 2], [5, Theorem 3], [6, Theorem 1] concerning linearly cushioned refinements. Furthermore, we shall define a new generalization of metric spaces and stratifiable spaces, called "elastic spaces," by introducing the notion of an "elastic base."

Definition 1. Let \( \mathcal{U} \) be a collection of subsets of a set \( X \), and let \( \mathfrak{R} \) be a relation on \( \mathcal{U} \) (i.e., \( \mathfrak{R} \subseteq \mathcal{U} \times \mathcal{U} \)). We shall often write \( U \mathfrak{R} V \) instead of \( (U, V) \in \mathfrak{R} \). The relation \( \mathfrak{R} \) is said to be a framed relation on \( \mathcal{U} \) (or a framing of \( \mathcal{U} \)) provided for every \( U, V \in \mathcal{U} \), if \( U \cap V \neq \emptyset \), then \( U \mathfrak{R} V \) or \( V \mathfrak{R} U \). We say \( \mathfrak{R} \) is a well-framed relation on \( \mathcal{U} \) provided \( \mathfrak{R} \) is a framing of \( \mathcal{U} \) and for every \( x \in X \), there exists an \( \mathfrak{R} \)-smallest \( U_x \in \mathcal{U} \) containing \( x \) (i.e., if \( x \in U, U \in \mathcal{U} \), and \( U \neq U_x \), then \( (U, U_x) \notin \mathfrak{R} \)).

Definition 2. A collection \( \mathcal{U} \) is said to be framed in a collection \( \mathcal{V} \) with frame map \( f: \mathcal{U} \rightarrow \mathcal{V} \), provided there exists a framed relation \( \mathfrak{R} \) on \( \mathcal{U} \) such that for every subcollection \( \mathcal{U}' \subseteq \mathcal{U} \) which has an \( \mathfrak{R} \)-upper bound (i.e., there exists \( U \in \mathcal{U} \) so that \( U \mathfrak{R} U' \) for every \( U' \in \mathcal{U}' \)) we have \( \text{cl}(\mathcal{U}' \cup U) \subseteq \bigcup f(\mathcal{U}') \). If in addition \( \mathfrak{R} \) is a well-framed relation on \( \mathcal{U} \), we say that \( \mathcal{U} \) is well-framed in \( \mathcal{V} \). Finally, if \( \mathcal{U} \) is framed in \( \mathcal{V} \) and \( \mathfrak{R} \) is also a transitive relation, then \( \mathcal{U} \) is called elastic in \( \mathcal{V} \), or an elastic refinement of \( \mathcal{V} \) when \( \mathcal{U} \) and \( \mathcal{V} \) are covers of \( X \).

Theorem 1. Let \( X \) be a regular space. A necessary and sufficient condition that \( X \) be paracompact is that every open cover of \( X \) have an open elastic refinement.

2. Proof of Theorem 1. The proof follows from the next two lemmas.
Lemma 1. Let $X$ be a regular space. A necessary and sufficient condition for $X$ to be paracompact is for every open cover of $X$ to have an open refinement which is well-framed in it.

Proof. We shall prove the sufficiency. The proof is similar to that of Theorem 1 in [6]. Let $\mathcal{U}$ be an open cover of $X$. Let $\mathcal{R}$ be an open refinement of $\mathcal{U}$ which is well-framed in $\mathcal{U}$ with respect to the well-framed relation $\mathcal{R}$ on $\mathcal{U}$ and frame map $f: \mathcal{U} \to \mathcal{U}$. Let $H_U = U - \bigcup \{ U' \in \mathcal{U} : U' \cap U \text{ and } U' \neq U \}$, and $\mathcal{F} = \{ H_U : U \in \mathcal{U} \}$. We now show that $\mathcal{F}$ is a cushioned refinement of $\mathcal{U}$ with cushion map $g: \mathcal{F} \to \mathcal{U}$ defined by $g(H_U) = f(U)$ (these terms are defined in [6]) and conclude that $X$ is paracompact by [3, Theorem 1.1, p. 309]. It is easy to see that $\mathcal{F}$ is a cover of $X$ since $\mathcal{R}$ is well-framed. It remains to show that $\mathcal{F}$ is cushioned in $\mathcal{U}$. Let $\mathcal{F}' \subset \mathcal{F}$, and suppose $x \in \bigcup g(\mathcal{F}')$. Let $U_x$ be an $\mathcal{R}$-smallest element of $\mathcal{R}$ containing $x$. Clearly $U_x$ is an open neighborhood of $x$ missing $H_{U_x}$ for all $U \neq U_x$ such that $U_x \cap U$. Further, $\mathcal{U}' = \{ U \in \mathcal{U} : U \cap U_x \text{ and } H_U \subseteq \mathcal{F}' \}$ has an $\mathcal{R}$-upper bound. Hence $\text{cl}(\bigcup \mathcal{U}') \subseteq \bigcup f(\mathcal{U}') \subseteq \bigcup g(\mathcal{F}')$ because $\mathcal{R}$ is framed in $\mathcal{U}$. Therefore, there exists an open neighborhood $N$ of $x$ missing $\text{cl}(\bigcup \mathcal{U}')$. Finally, if $U$ is not $\mathcal{R}$-related to $U_x$, then $U \cap U_x = \emptyset$ since $\mathcal{R}$ is a framing of $\mathcal{U}$. Thus, $U_x \cap N$ is an open neighborhood of $x$ missing $\bigcup \mathcal{F}'$, and we have $\text{cl}(\bigcup \mathcal{F}') \subseteq \bigcup g(\mathcal{F}')$. This completes the proof.

Lemma 2. Let $\mathcal{U}$ be a cover of a set $X$, and let $\leq$ be a transitive relation which is a framing of $\mathcal{U}$. Then there exists a well-framed relation $\mathcal{R}$ on $\mathcal{U}$ so that every subset of $\mathcal{U}$ with an $\mathcal{R}$-upper bound has a $\leq$-upper bound.

Proof. Let $\leq *$ be a well-order on an index set for $\mathcal{U}$ such that $\mathcal{U} = \{ U_0, U_1, \ldots, U_\alpha, \ldots : \alpha < * \eta \}$. Let $\Delta [\alpha] = \{ U \in \mathcal{U} : U \leq U_\alpha \}$ for all $\alpha < * \eta$. Well-order $\Delta [0]$ in any manner and denote the well-order by $\leq \leq^\alpha$. Suppose a reflexive and antisymmetric relation $\leq \leq^\beta$ has been defined on $\mathcal{U} \{ \Delta [\gamma] : \gamma \leq^* \beta \}$ for all $\beta <^* \alpha$ in such a way that $\leq \leq^\alpha$ is an extension of $\leq \leq^\beta$ whenever $\delta \leq^* \beta$. Put $\Delta' [\alpha] = \Delta [\alpha] - \{ \Delta [\beta] : \beta <^* \alpha \}$; well-order $\Delta' [\alpha]$ and denote the order by $\leq \leq^{\alpha \ast}$. Also define $U \leq^\alpha U'$ if $U' \in \Delta' [\alpha]$ and $U \in \Delta [\alpha] \cap \{ \Delta [\beta] : \beta <^* \alpha \}$. Let $\leq \leq$ be the relation on $\mathcal{U} \{ \Delta [\beta] : \beta \leq^* \alpha \}$ generated by $\leq \leq^\beta$ and $\leq^{\alpha \ast}$. Finally, let $\mathcal{R}$ be the reflexive and antisymmetric relation generated by $\leq \leq^{\alpha \ast}$. First we shall show that $\mathcal{R}$ is a framing of $\mathcal{U}$. If $U, U' \in \mathcal{U}$ and $U \cap U' \neq \emptyset$, then either $U \leq U'$ or $U' \leq U$ since $\leq \leq$ is a framing of $\mathcal{U}$.

2 The authors would like to thank E. Michael for some helpful suggestions concerning this result.
Suppose that $U \subseteq U'$. Let $\alpha_0$ be the first index such that $U' \in \Lambda [\alpha_0]$. Since $\subseteq$ is transitive, we have $U \in \Lambda [\alpha_0]$, and thus by the definition of $\alpha$ we know $U \not\subseteq U'$.

Next we show that if $\mathcal{U}$ has an $\mathfrak{R}$-upper bound, then $\mathcal{U}$ has a $\subseteq$-upper bound. To do this we first note that if $U \not\subseteq U'$, then for the first index $\alpha$ such that $U, U' \in U \{ \Lambda [\beta] : \beta \leq \alpha \}$ we have $U \subseteq \alpha U'$, $U' \subseteq \alpha U'$, and $U \in \Lambda [\alpha]$. To see this, let $\gamma$ be the first index such that $U \subseteq \gamma U'$. Since $\subseteq \leq \gamma$ is a relation on $U \{ \Lambda [\beta] : \beta \leq \gamma \}$ we know $\alpha \leq \gamma$. If $\alpha < \gamma$, then $U, U' \not\subseteq \gamma U'$ so $U, U'$ are not related by $\leq \gamma$. By the definition of $\leq \gamma$, we must have $U \subseteq \gamma U'$ for some $\beta \leq \gamma$, but this contradicts the definition of $\gamma$. Thus, $\alpha = \gamma$, and $U \subseteq \alpha U'$.

Further, not both of $U$ and $U'$ are in $U \{ \Lambda [\beta] : \beta < \delta \}$ for any $\delta < \alpha$, so $U$ and $U'$ are not related by $\leq \delta$. Thus, $U \subseteq \delta U'$, from which it follows that $U' \in \Lambda [\alpha]$ and $U \in \Lambda [\alpha]$. Now suppose $\mathcal{U}$ is a subcollection of $\mathcal{U}$ which has an $\mathfrak{R}$-upper bound. Let $U'$ be an $\mathfrak{R}$-upper bound of $\mathcal{U}$. Let $\alpha_0$ be the first index such that $U' \in \Lambda [\alpha_0]$. We now show that $\mathcal{U}$ has $U_0$ for $\subseteq$-upper bound. Let $U \in \mathcal{U}$ and let $\alpha_1$ be the first index such that $U \in \Lambda [\alpha_1]$. If $\alpha_1 \leq \alpha_0$, then $\alpha_0$ is the first index such that $U, U' \in U \{ \Lambda [\beta] : \beta \leq \alpha_0 \}$. Hence $U \not\subseteq U'$ implies $U, U' \in \Lambda [\alpha_0]$ as noted above. In particular $U \subseteq \alpha_0$ by definition of $\Lambda [\alpha_0]$. If $\alpha_0 < \alpha_1$ then $\alpha_1$ is the first index such that $U, U' \in \Lambda [\beta] : \beta \leq \alpha_1$. Hence $U \not\subseteq U'$ implies $U' \in \Lambda [\alpha_1]$, but this contradicts the fact that $U' \in \Lambda [\alpha_0]$.

Finally we show that every nonempty subset $\mathcal{U}'$ of $\mathcal{U}$ has an $\mathfrak{R}$-smallest element. Let $\alpha$ be the first index such that $\mathcal{U}' \cap \mathcal{U} [\alpha] \neq \emptyset$. Then $\mathcal{U}' \cap \mathcal{U} [\alpha] \subseteq \mathcal{U} [\alpha]$. Since $(\Lambda [\alpha], \leq)$ is a well-ordered subset of $\mathcal{U}$, there exists a $\leq$-first element of $\mathcal{U}' \cap \Lambda [\alpha]$ which is an $\mathfrak{R}$-smallest element of $\mathcal{U}'$. Thus $\mathfrak{R}$ is well-framed, and this completes the proof.

**Proof of Theorem 1.** We need only prove the sufficiency. Let $\mathcal{U}$ be an open cover of $X$, and let $\mathcal{W}$ be an open elastic refinement of $\mathcal{U}$. By Lemma 2, it is easy to see that there is a well-framed relation on $\mathcal{W}$, and that $\mathcal{W}$ is well-framed in $\mathcal{U}$. Hence $X$ is paracompact by Lemma 1.

### 3. Elastic spaces

According to J. G. Ceder [2], a collection $P$ of ordered pairs $P = (P_1, P_2)$ of subsets of a space $X$ is called a *pair base* for $X$ provided that $P_1$ is open for all $P \in P$ and that for every $x \in X$ and for every open set $U$ containing $x$, there exists a $P \in P$ such that $x \in P_1 \subseteq P_2 \subseteq U$. Further, he called a $T_1$-space an $M_1$-space (renamed *stratifiable space* by C. J. R. Borges [1]) provided it has a $\sigma$-cushioned pair base $P$. A pair base $P$ is said to be $\sigma$-cushioned provided $P$
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\[ \bigcup_{n=1}^{\infty} P_n \], and for every \( n \) and every \( P_n' \subset P_n \) we have \( \text{cl}(\bigcup \{ P_1: P \in P_n' \}) \subset \bigcup \{ P_2: P \in P_n' \} \).

**Definition 3.** A pair base \( P \) for a space \( X \) is said to be an *elastic base* if there is a framing of \( P_1 = \{ P_1: P = (P_1, P_2) \in P \} \) such that \( P_1 \) is elastic in \( P_2 = \{ P_2: P = (P_1, P_2) \in P \} \) with respect to the map \( f(P_1) = P_2 \). A \( T_1 \)-space with an elastic base is called an *elastic space*.

**Theorem 2.** Every subspace of an elastic space is an elastic space. Every metrizable space, and more generally every stratifiable space, is an elastic space. Every elastic space is paracompact.

**Proof.** The first statement is obvious. Let \( X \) be a stratifiable space with a \( \sigma \)-cushioned pair base \( P = \bigcup_{n=1}^{\infty} P_n \). We may assume that \( \{ P_n: n = 1, 2, \ldots \} \) is a partition of \( P \). Let \( \leq_n \) be a well-order on \( P_n \) for each \( n \), and define a well-order \( \leq \) on \( P \) as follows: For \( P, P' \in P \) we say \( P \leq P' \) if and only if either (1) \( P, P' \) are in the same \( P_n \) and \( P_n \leq P' \), or (2) \( P \in P_n, P' \in P_m, \) and \( n < m \). Then \( \leq \) obviously is an elastic base. Since an elastic space is regular, it follows from Theorem 1 that every elastic space is paracompact.

**Example.** (An elastic space which is not a stratifiable space.) Let \( X = [0, \Omega] \) be the set of ordinals less than or equal to the first uncountable ordinal. Let the topology on \( X \) be the weakest topology stronger than the order topology for which every point is isolated except \( \Omega \). Construct an elastic base for \( X \) as follows. Let \( U_\alpha = (\alpha, \Omega] \) for all \( \alpha < \Omega \), and let \( P' = \{ (U_\alpha, U_\alpha): \alpha \in [0, \Omega) \} \) and order \( P' \) by the usual order on the index set \( [0, \Omega) \). Let \( W_\alpha = \{ \alpha \} \) for all \( \alpha < \Omega \), and let \( P'' = \{ (W_\alpha, W_\alpha): \alpha \in [0, \Omega) \} \) and order \( P'' \) by the usual order on the index set \( [0, \Omega) \). Finally, set \( P = P'' \cup P'' \) and order \( P \) so that every element of \( P' \) precedes every element of \( P'' \). Then \( P \) is an elastic base for \( X \), so \( X \) is an elastic space. Clearly, \( X \) is not stratifiable because the closed set \( \{ \Omega \} \) is not a \( G_\delta \) in \( X \) (see [2, Theorem 2.2, p. 106]).

**Conjecture.** Every closed continuous image of an elastic space is an elastic space.

**References**


\(^3\) Tamano called this statement a theorem in his manuscript, but he did not give a complete proof; so it is stated here as a conjecture.


Texas Christian University, Fort Worth, Texas 76129

University of North Carolina, Chapel Hill, North Carolina 27514