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A FIXED POINT THEOREM FOR MANIFOLDS

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ABSTRACT. A Lefschetz type fixed point theorem is proved extending a recent theorem by Robert F. Brown. It deals with compact maps of the form $f:(M-U, X)\to(M, M-U)$, where M is an n-manifold, X is an $(n-2)$-connected ANR which is closed in M and U is an unbounded component of $M-U$. The map f defines maps $u:M-U\to M-U$ and $v:M\to M$; the Lefschetz numbers of u and v are defined and are shown to be equal; and if this number is nonzero then f has a fixed point.

Robert F. Brown [2] proved a generalization of the Brouwer fixed point theorem by making use of a retraction theorem of Bing [1]. A special case of Bing's retraction theorem was used earlier by Henderson and Livesay [5] to prove a theorem which is now a special case of the Brown theorem.

The purpose of this note is to prove a theorem which extends Brown's result in a similar sense as the Lefschetz fixed point theorem extends the Brouwer fixed point theorem. We use the ideas and results of [6].

Let V be a vector space over a field K. An endomorphism $f: V\to V$ will be said to be of finite type if there exists an integer m such that $f^m V$ is finite dimensional. In this case, according to Definition (2.3) of [6], the trace tr $f\in K$ of f is defined.

If now $V=\{V_n\}$ is a graded vector space and $f: V\to V$ is an endomorphism of degree zero, then f is said to be a Lefschetz endomorphism if each $f_n: V_n\to V_n$, is of finite type and all but a finite number of them are zero. In this case the Lefschetz number $\Lambda f =\sum_n (-1)^n \operatorname{tr} f_n$ is defined.

In order to define the Lefschetz number Λf of a map $f:X\to X$ of a topological space X for the purpose of this paper, we may use any functor H_* from the topological category to the category of graded vector spaces (over K) and homomorphisms of degree zero, such that H_* satisfies the homotopy axiom, the dimension axiom, and agrees with the usual homology on the category of compact polyhedra. Thus H_* may be the singular homology or the Čech homology, for instance.

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Let X be a topological space, $A \subset X$ and $f: A \to X$. Then the fixed point set of f is $F_f = \{x \in A \mid fx = x\}$. A map $f: X \to X$ is said to be a Lefschetz map provided that:

(1°) The induced homomorphism $f_* = H_* f: H_* X \to H_* X$ is a Lefschetz endomorphism, in which case the Lefschetz number Λf of f is defined to be $\Lambda f = \Lambda(f_*)$;

(2°) The condition $\Lambda f \neq 0$ implies $F_f \neq \emptyset$.

A topological space X is said to be a Λ-space (see [6, (2.6)]) if every compact map $f: X \to X$ is a Lefschetz map. It was shown in [6, (5.2)], that ANR's are Λ-spaces (see also [3]).

The following theorem is the main result of this note:

THEOREM 1. *Let M be an n-manifold (with or without boundary), let X be an $(n-2)$-connected ANR imbedded as a closed subset of M and let U be a component of $M - X$ whose closure is not compact. Let $f: (M-U, X) \to (M, M-U)$ be a compact map and let $f': X \to M-U$ denote the map defined by the restriction of f. Then there exist Lefschetz maps $u: M-U \to M-U$ and $v: M \to M$ such that u is an extension of f', v is an extension of f, $\Lambda u = \Lambda v$ and $F_v \subset F_u \cap F_f$.*

In particular, if $\Lambda v \neq 0$ then f has a fixed point.

COROLLARY (BROWN [2]). *Let M, X, U and f be as in Theorem* 1 *and assume that M is acyclic. Then f has a fixed point.*

For, in this case, the Lefschetz number of $v: M \to M$ is equal to one.

We shall first prove

THEOREM 2. *Suppose that a map $f: X \to X$ can be factored through a Λ-space Y:*

$$\begin{tikzcd} & Y \arrow[dr, "h"] & \\ X \arrow[ur, "g"] \arrow[rr, "f"'] & & X \end{tikzcd} \qquad f = h \circ g$$

such that: either (a) *g is compact; or* (b) *Y is Hausdorff and h is compact. Then f is a Lefschetz map, $\Lambda f = \Lambda(g \circ h)$ and $h(F_{g \circ h}) \subset F_f$.*

(Compare [6, (3.1)].)

PROOF. Condition (a) or (b) implies that $g \circ h: Y \to Y$ is compact. Since Y is a Λ-space, $(g \circ h)_* = g_* \circ h_*: H_* Y \to H_* Y$ is a Lefschetz endomorphism; moreover, $\operatorname{tr} f_* = \operatorname{tr}(h_* \circ g_*) = \operatorname{tr}(g_* \circ h_*)$ (see [6, (2.4)]). It follows that f_* is a Lefschetz endomorphism and $\Lambda f_* = \Lambda(g \circ h)_*$, so $\Lambda f = \Lambda(g \circ h)$. If $y_0 \in F_{g \circ h}$ then clearly $hy_0 \in F_f$. Thus if $\Lambda f \neq 0$ then $\Lambda(g \circ h) \neq 0$; consequently $F_{g \circ h} \neq 0$ and hence $F_f \neq 0$.

THEOREM 3. *A manifold (with or without boundary) is a Λ-space.*

PROOF. If M is metrizable then it is a local ANR, and hence an ANR by [4]. Consequently, in this case, it is a Λ-space by [6, (5.2)]. The general case will follow from

LEMMA. *Every compact subspace of an n-manifold M is contained in a metrizable manifold (with a countable basis).*

PROOF. Let $K\subset M$ be compact. Then K can be covered by closed n-disks $D_1, \cdots, D_s$ whose interiors $B_i=\operatorname{Int} D_i$, $i=1, \cdots, s$, cover K. Thus K is metrizable with a countable basis and therefore $N=B_1\cup \cdots \cup B_s$ is a manifold with these properties containing K.

To prove the general case of Theorem 3, suppose that $f: M\to M$ is a compact map. Then, by the Lemma, $K=\overline{fM}$ is contained in a metrizable manifold N. Thus we obtain a factorization

$$\begin{tikzcd} & N \arrow[dr, "h"] & \\ M \arrow[ur, "g"] \arrow[rr, "f"] & & M \end{tikzcd}$$

where g is defined by f and h is the inclusion. Since g is compact and N is a Λ-space, we may apply Theorem 2 to conclude that f is a Lefschetz map and that $\Lambda f=\Lambda(g\circ h)$.

PROOF OF THEOREM 1. The remaining part of the proof of Theorem 1 is analogous to that of [2]. By Bing's Retraction Theorem [1], there exists a retraction $X\cup U\to X$ which extends to a retraction $r: M\to M-U$. We have the factorization

$$\begin{tikzcd} & M \arrow[dr, "r"] & \\ M-U \arrow[ur, "f"] \arrow[rr, "r\circ f"] & & M-U \end{tikzcd}$$

Set $u=r\circ f: M-U\to M-U$, $v=f\circ r: M\to M$. By Theorem 3, M is a Λ-space; and by Theorem 2, $\Lambda u=\Lambda v$. Suppose that $x_0\in F_v$, i.e., $frx_0=x_0$. Then $y_0=rx_0\in F_u$. If we suppose that $x_0\in U$, then $y_0=rx_0\in X$, so $fy_0=x_0\in M-U$, since $fX\subset M-U$, which is a contradiction. It follows that $x_0\in M-U$ and hence $rx_0=x_0$. Thus $x_0\in F_u$ and $x_0\in F_f$.

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