A FIXED POINT THEOREM FOR MANIFOLDS

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Abstract. A Lefschetz type fixed point theorem is proved extending a recent theorem by Robert F. Brown. It deals with compact maps of the form \( f: (M - U, X) \to (M, M - U) \), where \( M \) is an \( n \)-manifold, \( X \) is an \((n-2)\)-connected ANR which is closed in \( M \) and \( U \) is an unbounded component of \( M - U \). The map \( f \) defines maps \( u: M - U \to M - U \) and \( v: M \to M \); the Lefschetz numbers of \( u \) and \( v \) are defined and are shown to be equal; and if this number is nonzero then \( f \) has a fixed point.

Robert F. Brown [2] proved a generalization of the Brouwer fixed point theorem by making use of a retraction theorem of Bing [1]. A special case of Bing's retraction theorem was used earlier by Henderson and Livesay [5] to prove a theorem which is now a special case of the Brown theorem.

The purpose of this note is to prove a theorem which extends Brown's result in a similar sense as the Lefschetz fixed point theorem extends the Brouwer fixed point theorem. We use the ideas and results of [6].

Let \( V \) be a vector space over a field \( K \). An endomorphism \( f: V \to V \) will be said to be of finite type if there exists an integer \( m \) such that \( f^m V \) is finite dimensional. In this case, according to Definition (2.3) of [6], the trace \( \text{tr} f \in K \) of \( f \) is defined.

If now \( V = \{ V_n \} \) is a graded vector space and \( f: V \to V \) is an endomorphism of degree zero, then \( f \) is said to be a Lefschetz endomorphism if each \( f_n: V_n \to V_n \), is of finite type and all but a finite number of them are zero. In this case the Lefschetz number \( \Lambda f = \sum (-1)^n \text{tr} f_n \) is defined.

In order to define the Lefschetz number \( \Lambda f \) of a map \( f: X \to X \) of a topological space \( X \) for the purpose of this paper, we may use any functor \( H_* \) from the topological category to the category of graded vector spaces (over \( K \) and homomorphisms of degree zero, such that \( H_* \) satisfies the homotopy axiom, the dimension axiom, and agrees with the usual homology on the category of compact polyhedra. Thus \( H_* \) may be the singular homology or the Čech homology, for instance.

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Let $X$ be a topological space, $A \subset X$ and $f : A \to X$. Then the fixed point set of $f$ is $F_f = \{ x \in A \mid fx = x \}$. A map $f : X \to X$ is said to be a Lefschetz map provided that:

1° The induced homomorphism $f_* = H_* f : H_* X \to H_* X$ is a Lefschetz endomorphism, in which case the Lefschetz number $\Lambda f$ of $f$ is defined to be $\Lambda f = \Lambda (f_*)$;

2° The condition $\Lambda f \neq 0$ implies $F_f \neq \varnothing$.

A topological space $X$ is said to be a $\Lambda$-space (see [6, (2.6)]) if every compact map $f : X \to X$ is a Lefschetz map. It was shown in [6, (5.2)], that ANR’s are $\Lambda$-spaces (see also [3]).

The following theorem is the main result of this note:

**Theorem 1.** Let $M$ be an $n$-manifold (with or without boundary), let $X$ be an $(n-2)$-connected ANR imbedded as a closed subset of $M$ and let $U$ be a component of $M - X$ whose closure is not compact. Let $f : (M - U, X) \to (M, M - U)$ be a compact map and let $f : X \to M - U$ denote the map defined by the restriction of $f$. Then there exist Lefschetz maps $u : M - U \to M - U$ and $v : M \to M$ such that $u$ is an extension of $f^\prime$, $v$ is an extension of $f$, $\Lambda u = \Lambda v$ and $F_v \subset F_u \cap F_f$.

In particular, if $\Lambda v \neq 0$ then $f$ has a fixed point.

**Corollary (Brown [2]).** Let $M$, $X$, $U$ and $f$ be as in Theorem 1 and assume that $M$ is acyclic. Then $f$ has a fixed point.

For, in this case, the Lefschetz number of $v : M \to M$ is equal to one.

We shall first prove

**Theorem 2.** Suppose that a map $f : X \to X$ can be factored through a $\Lambda$-space $Y$:

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
 \searrow{h} & \downarrow{f} & \nearrow{v} \\
X & \xrightarrow{f} & X
\end{array}
\]

such that: either (a) $g$ is compact; or (b) $Y$ is Hausdorff and $h$ is compact. Then $f$ is a Lefschetz map, $\Lambda f = \Lambda (g \circ h)$ and $h(F_{v \circ h}) \subset F_f$.

(Compare [6, (3.1)].)

**Proof.** Condition (a) or (b) implies that $g \circ h : Y \to Y$ is compact. Since $Y$ is a $\Lambda$-space, $(g \circ h)_* = g_* \circ h_* : H_* Y \to H_* Y$ is a Lefschetz endomorphism; moreover, $\text{tr } f_* = \text{tr } (h_* \circ g_*) = \text{tr } (g_* \circ h_*)$ (see [6, (2.4)]). It follows that $f_*$ is a Lefschetz endomorphism and $\Lambda f_* = \Lambda (g \circ h)_*$, so $\Lambda f = \Lambda (g \circ h)$. If $y_0 \in F_{v \circ h}$ then clearly $h y_0 \in F_f$. Thus if $\Lambda f \neq 0$ then $\Lambda (g \circ h) \neq 0$; consequently $F_{v \circ h} \neq 0$ and hence $F_f \neq 0$. 

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Theorem 3. A manifold (with or without boundary) is a $\Lambda$-space.

Proof. If $M$ is metrizable then it is a local ANR, and hence an ANR by [4]. Consequently, in this case, it is a $\Lambda$-space by [6, (5.2)]. The general case will follow from

Lemma. Every compact subspace of an $n$-manifold $M$ is contained in a metrizable manifold (with a countable basis).

Proof. Let $K \subset M$ be compact. Then $K$ can be covered by closed $n$-disks $D_1, \ldots, D_s$ whose interiors $B_i = \text{Int} D_i$, $i = 1, \ldots, s$, cover $K$. Thus $K$ is metrizable with a countable basis and therefore $N = B_1 \cup \cdots \cup B_s$ is a manifold with these properties containing $K$.

To prove the general case of Theorem 3, suppose that $f : M \to M$ is a compact map. Then, by the Lemma, $K = fM$ is contained in a metrizable manifold $N$. Thus we obtain a factorization

\[
\begin{array}{c}
N \\
g \\
f \downarrow \\
M \\
h
\end{array}
\]

where $g$ is defined by $f$ and $h$ is the inclusion. Since $g$ is compact and $N$ is a $\Lambda$-space, we may apply Theorem 2 to conclude that $f$ is a Lefschetz map and that $\Lambda f = \Lambda (g \circ h)$.

Proof of Theorem 1. The remaining part of the proof of Theorem 1 is analogous to that of [2]. By Bing's Retraction Theorem [1], there exists a retraction $X \cup U \to X$ which extends to a retraction $r : M \to M - U$. We have the factorization

\[
\begin{array}{c}
M \\
\downarrow f \\
r \circ f \downarrow \\
M - U \\
\downarrow v \\
M - U
\end{array}
\]

Set $u = r \circ f : M - U \to M - U$, $v = f \circ r : M \to M$. By Theorem 3, $M$ is a $\Lambda$-space; and by Theorem 2, $\Lambda u = \Lambda v$. Suppose that $x_0 \in F_u$, i.e., $frx_0 = x_0$. Then $y_0 = rx_0 \in F_u$. If we suppose that $x_0 \in U$, then $y_0 = rx_0 \in X$, so $f y_0 = x_0 \in M - U$, since $f X \subset M - U$, which is a contradiction. It follows that $x_0 \in M - U$ and hence $rx_0 = x_0$. Thus $x_0 \in F_u$ and $x_0 \in F_f$.

References


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