ON AN ASYMPTOTIC PROPERTY OF A VOLterra INTEGRAL EQUATION

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Abstract. It is proved that if \( q(t-s) \) is bounded and \( f(t, x) \) is "small," the solutions of the integral equation \( x(t) = a(t) + \int_0^t q(t-s)f(s, x(s)) \, ds \) satisfies the conditions \( x(t) = h(t) + \rho(t)a(t) \), \( \lim_{t \to \infty} a(t) = a \) constant where \( \rho(t) \) is a nonsingular diagonal matrix chosen in such a way that \( \rho(t)h(t) \) is bounded. The results are extended to the more general integral equation \( x(t) = h(t) + \int_0^t F(t, s, x(s)) \, ds \) and contain, in particular, some results on the boundedness, asymptotic behavior and existence of nonoscillatory solution of differential equations.

Consider the system of Volterra integral equations

\[
(1) \quad x(t) = h(t) + \int_0^t q(t-\tau)f(\tau, x(\tau)) \, d\tau + \int_0^t g(t-\tau) \, d\tau
\]

where \( h = (h_1, h_2, \ldots, h_n) \), \( f = (f_1, f_2, \ldots, f_n) \), \( g = (g_1, g_2, \ldots, g_n) \) are column vectors in \( \mathbb{R}^n \), \( q = (q_{ij}) \) is an \( n \times n \) matrix and \( \mathbb{R}^n \) is Euclidean \( n \)-dimensional space. In what follows \( \lim_{t \to \infty} \) means limit almost everywhere.

We assume the hypotheses
\[(H_1) \quad |q(t)|, |g(t)| \in L(0, C], 0 < C < \infty, \]
\[(H_2) \quad h(t) \text{ is continuous for } 0 \leq t < \infty, \]
\[(H_3) f(t, x) \text{ is continuous for } 0 \leq t < \infty, |x| < \infty. \]

These conditions guarantee the local existence of continuous solutions and continuability of each solution so long as they remain bounded [6, p. 324]. We show here that if \( f, g \) and \( q \) satisfy some additional hypotheses then the solutions of (1) satisfy the condition \( x_1(t) = h_1(t) + a_1(t) \rho_1(t) \) with \( \lim_{t \to \infty} a_1(t) = a_1 \) or \( a_1(t) \) bounded, where \( \rho_1(t) > 0 \) is chosen in such a way that \( h_1(t)/\rho_1(t) \) is bounded. Some similar results were obtained by Strauss [8] under different hypotheses.

We need the following lemma.

**Lemma 1.** Let \( E \) be a measurable set of points of any number of dimensions of finite or infinite measure and let \( f(t, s) \) be summable in \( E \) for
values of \( t \) in \([0, \infty)\). Assume that there exists a summable nonnegative function \( \phi(s) \) such that \(|f(t, s)| \leq \phi(s)\) for almost all values of \( s \) in \( E \) and all values of \( t \) in \((i_0, \infty)\). Then if \( \lim_{t \to \infty} f(t, s) \) exists for all (or almost all) values of \( s \) in \( E \) we have

\[
\lim_{t \to \infty} \int_E f(t, s) ds = \int_E \lim_{t \to \infty} f(t, s) ds.
\]

For a proof of Lemma 1, see [4, p. 322]. The following lemma which is proved in a more general form in [6, p. 326] gives conditions for the global existence of solutions of (1).

**Lemma 2.** Let \( \omega(t, r) \) be continuous in \((t, r)\) for \(0 \leq t < \infty, 0 \leq r < \infty, \) and nondecreasing in \( r \) for each fixed \( t \). For some \( b < \infty \) let \( r_M \) be the maximum solution of

\[
(2) \quad \dot{r} = K \omega(t, r), \quad 0 \leq t < b, \quad \dot{r}(0) = K,
\]

and suppose that \(|h(t)| \leq K, |q(t)| \leq K\) and \(|f(t, x)| \leq \omega(t, |x|), 0 \leq t < \infty, |x| < \infty\). Then if \( \phi \) is a solution of (1) \( \phi \) can be continued to the right as far as \( r_M \) exists and

\[
|\phi(t)| \leq r_M(t), \quad 0 \leq t < b.
\]

In particular, if \( r_M(t) \) exists for \(0 \leq t < \infty\) one has global existence for solutions of (1).

**Lemma 3.** Let \( a_i \geq 0, b_i \geq 0, r_i \geq 0 \) and \( r = \max_i r_i, i = 1, 2, \ldots, n \), if \( b_i > 1 \) for some \( i \) then

\[
\sum_{i=1}^{n} a_i b_i ^{r_i} \leq \left[ \sum_{i=1}^{n} a_i \right] \left[ \sum_{i=1}^{n} b_i \right].
\]

Let \( \rho_i(t) > 0, i = 1, 2, \ldots, n \), be continuous functions for \(0 \leq t < \infty\) in such a way that \( h_i(t)/\rho_i(t), i = 1, 2, \ldots, n \), are bounded for \(0 \leq t < \infty\).

We assume also the hypotheses

\begin{itemize}
  \item \( (H_4) \) \(|f_i(t, x)| \leq \sum_{j=1}^{n} \epsilon_{ij}(t)|x_j|^r, i = 1, 2, \ldots, n \), where \( \epsilon_{ij}(t) \) is non-negative, \( r_j > 0, j = 1, 2, \ldots, n \), \(|x| < H \leq \infty \) and
  \( \int_{0}^{H} \epsilon_{ij}(t) [\rho_j(t)] r_j [\rho_i(t)]^{-1} dt < \infty. \)
  \item \( (H_6) \) \(|g_{ij}(t) [\rho_i(t)]^{-1} dt < \infty. \)
  \item \( (H_6) \) \( g_{ij}(t-\tau) / \rho_j(\tau) / \rho_i(t) \) are bounded for \(0 \leq \tau \leq t < \infty, i, j = 1, 2, \ldots, n. \)
  \item \( (H_6') \) \( g_{ij}(t-\tau) / \rho_j(\tau) / \rho_i(t) \) are bounded for \(0 \leq \tau \leq t < \infty \) and \( \lim_{t \to \infty} (g_{ij}(t-\tau) / \rho_i(t)) \) exists for almost all values of \( \tau \).
\end{itemize}
Theorem 1. Assume with respect to equation (1) the hypotheses $(H_4)$, $(H_5)$ and $(H_6)$ and let $r$ be the maximum of the $r_i$, $i = 1, 2, \ldots, n$. Then

A. If $r > 1$ there exists $c > 0$ depending on $r$ and $\epsilon$, $i, j = 1, 2, \ldots, n$, such that $|h_i(t)/\rho_i(t)| \leq c$, $i = 1, 2, \ldots, n$, implies that the solutions of (1) exist in $[0, \infty)$ and satisfy the condition

$$x_i(t) = h_i(t) + a_i(t)\rho_i(t)$$

where $a_i(t)$ is bounded. If instead of condition $(H_6)$, (1) satisfies $(H'6)$ then $\lim_{t \to \infty} a_i(t) = a_i$ constant. Furthermore, if $\lim_{t \to \infty} (h_i(t)/\rho_i(t)) = h_i$ constant

$$\lim_{t \to \infty} \frac{x_i(t)}{\rho_i(t)} = b_i \text{ constant.}$$

B. If $r \leq 1$, $c$ can be chosen arbitrarily.

Proof.

$$x_i(t) = h_i(t) + \int_0^t \sum_{r=1}^n q_{ir}(t-s)f_r(s, x(s))\,ds + \int_0^t \sum_{j=1}^n q_{ij}(t-s)g_j(s)\,ds$$

or

$$\frac{|x_i(t)|}{\rho_i(t)} \leq k_i + \int_0^t \sum_{r=1}^n \frac{|q_{ir}(t-s)|}{\rho_i(t)} \sum_{j=1}^n \epsilon_{rj}(s) [\rho_j(s)] r^j \rho_r^{-1}(s) \frac{|x_j(s)| r^j ds}{[\rho_j(s)] r^j}$$

$$+ \int_0^t \sum_{j=1}^n \frac{|q_{ij}(t-s)|}{\rho_i(t)} \frac{|x_j(s)| r^j ds}{[\rho_j(s)] r^j}$$

$$\leq k_i + \int_0^t \sum_{r=1}^n c_r \sum_{j=1}^n \epsilon_{rj}(s) [\rho_j(s)] r^j \rho_r^{-1}(s) \frac{|x_j(s)| r^j ds}{[\rho_j(s)] r^j}$$

$$+ \int_0^t \sum_{j=1}^n c_j \frac{|g_j(s)|}{\rho_j(s)} \rho_j(s)^{-1} \, ds$$

$$\leq K_i + k \int_0^t \left( \sum_{rj} \epsilon_{rj}(s) [\rho_j(s)] r^j \rho_r^{-1}(s) \right) \sum_{i=1}^n \frac{|x_i(s)| r^i ds}{[\rho_i(s)] r^i}$$

where $K_i = k_i + \int_0^\infty \sum_{j=1}^n c_j \frac{|g_j(s)|}{\rho_j(s)} \rho_j(s)^{-1} \, ds$. If $|x_i(s)|/\rho_i(s) \leq 1$ for every $i = 1, 2, \ldots, n$, $\sum_{i=1}^n (|x_i(s)|/\rho_i(s)) r_i$ is bounded. If for some $i$, $|x_i(s)|/\rho_i(s) \leq 1$, then

$$\sum_{i=1}^n \frac{|x_i(s)| r^i}{[\rho_i(s)] r^i} \leq n \left( \sum_{i=1}^n \frac{|x_i(s)|}{\rho_i(s)} \right)^r$$

by Lemma 3, hence

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\[ \sum_{i=1}^{n} \left| \frac{x_i(t)}{\rho_i(t)} \right| \leq M + M \int_0^t \left( \sum_{r \neq j} \epsilon_{rj}(s) \left[ \rho_j(s) \right]^{r} \rho_r(s)^{-1} \right) \left( \sum_{i=1}^{n} \left| \frac{x_i(s)}{\rho_i(s)} \right|^r \right) ds. \]

By Lemma 2, \( \sum_{i=1}^{n} \left( \left| \frac{x_i(t)}{\rho_i(t)} \right| \right) \leq z(t) \) where \( z(t) \) is the maximal solution of the differential equation

\[ \dot{z} = M \left( \sum_{r \neq j} \epsilon_{rj}(s) \left[ \rho_j(s) \right]^{r} \rho_r(s)^{-1} \right) z^r. \]

The solution of (3) is

\[ z(t) = z(0) \exp \int_0^t \sum_{r \neq j} \epsilon_{rj}(s) \left[ \rho_j(s) \right]^{r} \rho_r(s)^{-1} ds \text{ if } r = 1, \]

\[ z(t)^{1-r} = z(0)^{1-r} + (1 - r) \exp M \int_0^t \sum_{r \neq j} \epsilon_{rj}(s) \left[ \rho_j(s) \right]^{r} \rho_r(s)^{-1} ds \text{ if } r > 1. \]

Then if \( r = 1 \) all solutions of (3) are bounded in \([0, \infty)\). If \( r > 1 \) choosing \( |z(0)| \leq c \) for \( c \) small enough the solutions of (3) are also bounded. Then by Lemma 2 the solutions of (1) exist globally.

Thus in the conditions of hypothesis \((H_6)\), \( x_i(t) = h_i(t) + \rho_i(t) a_i(t) \) where

\[ a_i(t) = \int_0^t \sum_{r \neq 1} \frac{q_{ir}(t-s)}{\rho_i(t)} f_r(s, x(s)) ds + \int_0^t \sum_{j=1}^{n} \frac{q_{ij}(t-s)}{\rho_i(t)} g_j(s) ds \]

is bounded and in the conditions of hypothesis \((H_4)\), by Lemma 1 \( \lim a_i(t) = a_i \) constant. The condition of convergence of the integral \( \int_0^t \epsilon_{ij}(t) \left[ \rho_j(t) \right]^{r} \left[ \rho_r(t) \right]^{-1} dt \) in hypothesis \((H_4)\) cannot be improved as is shown by the simple example

\[ x_1(t) = 2t + 1 + \int_0^t 2(t-s)(s+1)^{-2} x_1(s) ds, \]

\[ x_2(t) = 2 + \int_0^t 2(s+1)^{-2} x_1(s) ds \]

which has the solution \((t+1)^2, 2(t+1)\) which does not satisfy Theorem 1. In this example, \( q_{11} = 1, q_{12} = t-s, q_{21} = 0, q_{22} = 1, \epsilon_{11}(t) = 0, \epsilon_{12}(t) = 0, \epsilon_{21}(t) = 2(t+1)^2, \rho_1(t) = 2(t+1), \rho_2(t) = 2. \)

The example above shows that Theorem 1 is not true even when \( \lim_{t \to \infty} \epsilon_{ij}(t) = 0 \). Strauss \([8]\) uses a hypothesis of convergence to zero of the \( \epsilon_{ij}(t) \) and obtains similar results but uses a different hypotheses for the kernel \( q(t-s) \).
**Remark 1.** It is quite obvious that Theorem 1 is also true when $q = q(t - s)$ is not of convolution type. Hypothesis (H₄) to (H₆) were motivated by anterior research on differential equations and are general enough to include implicitly Theorem 2.1 of [3], Theorem 1 of [9] and several results on the boundedness and asymptotic behavior of differential equations, see [2, pp. 37 and 42] for references. The hypotheses imposed to the kernel $q$ can be weakened and are better stated in the context of the more general equation

\[(4) \quad x(t) = h(t) + \int_0^t F(t, s, x(s))ds\]

where $F = (F₁, F₂, \ldots, Fₙ)$ is continuous in $t$ and $x$ and $|F|$ is locally Lebesgue integrable. Theorems on the existence of continuous solutions, continuity of solutions and comparison theorems for (4) can be found in [5], [6] and [7].

We assume also the hypothesis

\[|Fₖ(t, s, x(s))| \leq \sum_{j=1}^{m} g_{ij}(s) |x_j(s)|^{r_j},\]

\[(H₇) \quad i = 1, 2, \ldots, n, \quad |x| < H \leq \infty,\]

where $g_{ij}(s)$ is nonnegative, $r_j \geq 0$, $j = 1, 2, \ldots, n$, and

\[\int_0^\infty g_{ij}(t)[\rho_i(t)]^{r_j}dt < \infty.\]

\[(H₇) \lim_{t \to \infty} (F_i(t, s, x(s))/\rho_i(t)) \text{ exist and is finite for almost all values of } s \text{ and } (H₇) \text{ is satisfied.}\]

Then conditions A and B of Theorem 1 are satisfied for equation (4), that is, the solutions of (4) satisfy $x_i(t) = h_i(t) + \rho_i(t)a_i(t)$ with $a_i(t)$ bounded in the conditions of hypothesis (H₇) and $\lim_{t \to \infty} a_i(t) = a_i$ in the conditions of hypothesis (H₇). Still more generally let $\rho = (\rho_i)$ be a diagonal matrix, $\rho_i(t) > 0$, $i = 1, 2, \ldots, n$, chosen in such a way that $|\rho(t)^{-1}h(t)| \leq K$. Suppose that $|\rho(t)^{-1}F(t, s, x(s))| \leq \omega(t, s, |\rho(s)^{-1}x(s)|)$ where $\omega(t, s, r)$ is monotone nondecreasing in $r$ for each fixed pair $(t, s)$. Then if the solutions of the integral equation $y(t) = K + \int_0^t \omega(t, s, y(s))ds$ are bounded, the solutions of (4) satisfy $x(t) = h(t) + \rho(t)a(t)$ with $a(t)$ bounded. If in addition $\lim_{t \to \infty} \rho(t)^{-1}F(t, s, x(s))$ exist and is finite then $\lim_{t \to \infty} a(t) = a$ const.

**Remark 2.** The following example shows that without hypothesis (H₇) we can not really guarantee that $\lim_{t \to \infty} a_i(t)$ exist. Consider the linear integral equation

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\[ x(t) = 1 + \frac{1}{1 + t} + \cos\left(1 - \frac{1}{1 + 1/(1 + t)}\right) \]

\[-\cos t \int_0^t \frac{1}{(1 + s)^2(1 + 1/(1 + s))^2} x(s) ds\]

which has the solution \( x(t) = 1 + 1/(1 + t) - \cos t(1/1 + 1/(1 + t)) + \frac{1}{2} \cos t. \) We can put \( p(t) = 1 + 1/(1 + t). \) Then

\[
\frac{x(t) - h(t)}{p(t)} = \frac{\cos t}{1 + 1/(1 + t)} \int_0^t \frac{1}{(s + 1)^2(1 + 1/(1 + s))^2} x(s) ds
\]

\[
= \frac{\cos t}{1 + 1/(1 + t)} \left[ \frac{1}{1 + 1/(1 + t)} - \frac{1}{2} \right]
\]

which does not have a limit when \( t \to \infty. \)

In an analogous manner the integral equation

\[
x(t) = 1 + \frac{1}{1 + t} + \frac{1}{1 + 1/(1 + t)} + \frac{1}{2}
\]

\[-\int_0^t \frac{1}{(1 + s)^2(1 + 1/(1 + s))^2} x(s) ds\]

shows that, in general, we can not have

\[
\lim_{t \to \infty} \frac{(x(t) - h(t))}{p(t)} = 0.
\]

**Remark 3.** If \( h(t) = \text{constant} \) in (1), then in the condition of hypothesis \((H_6), \lim_{t \to \infty} x(t) = C \text{ constant} \) if \( x(t) \) is bounded in \([0, \infty)\). Also the solutions of (1) tend to a point of equilibrium and equation (1) has asymptotic equilibrium in the sense defined by Wintner for differential equations (see [2, p. 43]).

**Remark 4.** As a simple application of Theorem 1 consider the equation

\[
y^{(n)} + f(t, y^{(1)}, \ldots, y^{(n-1)}) = g(t)
\]

which is equivalent to the integral equation

\[
y^{(k)}(t) = \theta_0 + \theta_1 t + \cdots + \theta_{n-k-2} t^{n-k-1} + \int_{t_0}^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} g(s) ds
\]

\[-\int_{t_0}^t \frac{(t-s)^{n-k-1}}{(n-k-1)!} f(s, y^{(1)}(s), \ldots, y^{(n)}(s)) ds,
\]

\[
K = 0, 1, \ldots, n - 1.
\]
If we choose $p_k(t) = t^{n-k-1}$, the kernel of (6) satisfies hypotheses $(H'_1)$ and $(H'_2)$. If $f$ and $g$ satisfy the hypotheses of Theorem 1 then (5) has a solution satisfying $\lim (y^{(k)}(t)/t^{n-k-1}) = A_K$. Since the integrals in (6) are convergent for $t_0$ sufficiently large, $A_K \neq 0$, $K = 1, 2, \ldots, n-1$, and we obtain Theorem 2.1 of [2] with weaker hypotheses which contains Theorem 1 of [9]. Brauer and Wong [1] treat the same problem from a different point of view and obtain the same results in a more general context.

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