PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, No. 1, April 1971

ARCS IN HYPERSPACES WHICH ARE NOT COMPACT

L. E. WARD, JR.[1]

ABSTRACT. It has been known for many years that if X is a metrizable continuum then 2^X (the space of closed subsets of X) and $C(X)$ (the subspace of connected members of 2^X) are arcwise connected. These results are due to Borsuk and Mazurkiewicz [1] and J. L. Kelley [2], respectively. Quite recently M. M. McWaters [6] extended these theorems to the case of continua which are not necessarily metrizable, using Koch's arc theorem for partially ordered spaces [3], [8]. In this note we prove these results for certain noncompact spaces by means of a simple generalization of Koch's arc theorem.

1. **Introduction.** Recall that if X is a topological space then 2^X denotes the space of nonempty closed subsets of X with the Vietoris topology [7]. That is, if $U_1, U_2, \cdots, U_n$ are subsets of X we write

$$\langle U_1, U_2, \cdots, U_n\rangle = \left\{ A \in 2^X : A \subset \bigcup_{i=1}^{n} \{U_i\} \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, 2, \cdots, n \right\}$$

and the family of all $\langle U_1, U_2, \cdots, U_n\rangle$ with U_i open is a base for the open sets. The subspace of all closed and connected sets is denoted $C(X)$.

In this note our principal result is a partial extension of the recent theorem of McWaters [6], that if X is an arbitrary continuum, then 2^X and $C(X)$ are arcwise connected.

THEOREM 1. *If X is a locally compact, locally connected and connected Hausdorff space, then 2^X is arcwise connected. If, in addition, X is a normal space then $C(X)$ is arcwise connected.*

At the end of the paper we give an example which shows that the hypothesis of local connectivity cannot be omitted.

It is a pleasure to express my appreciation to Professor Sam B. Nadler, Jr. who rekindled my interest in hyperspaces through several conversations and a lively correspondence.

Received by the editors January 5, 1970.

AMS 1969 subject classifications. Primary 5455; Secondary 0620, 5456.

Key words and phrases. Hyperspace, arc, partially ordered space.

[1] This research was supported by a grant from the National Science Foundation.

Copyright © 1971, American Mathematical Society

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

2. **A theorem on partially ordered spaces.** A *partially ordered space* is a topological space S with a partial order Γ which is a closed subset of $S\times S$. We treat the symbols $y\leqq x$, $y\in\Gamma x$, $x\in y\Gamma$ and $(y, x)\in\Gamma$ as synonyms. Of course $y<x$ means that $y\leqq x$ and $y\neq x$. A *zero* of a partially ordered space S is an element $0\in S$ such that $0\Gamma=S$.

McWaters' proof of arcwise connectivity depended on showing that if X is a continuum then 2^X and $C(X)$ can be regarded as partially ordered spaces which satisfy the following form of Koch's arc theorem [**3**], [**8**]: *if S is a compact partially ordered space with zero and if Γx is connected for each $x\in S$, then each nonzero element of S is the supremum of an order arc containing the zero.* For the applications of interest to us, a stronger version of Koch's theorem is required.

THEOREM 2. *Let S be a partially ordered space with zero, and suppose that for each nonzero $x\in S$ there exists $y<x$ such that if $y\leqq t\leqq x$ then $y\Gamma\cap\Gamma t$ is a continuum. If each nonempty chain of S has an infimum then S is arcwise connected.*

PROOF. Let 0 be the zero of S and let $x\in S-\{0\}$. By hypothesis there exists $y<x$ such that $y\Gamma\cap\Gamma x$ is compact and $y\Gamma\cap\Gamma t$ is connected for each $t\in y\Gamma\cap\Gamma x$. Therefore, $y\Gamma\cap\Gamma x$ satisfies the hypotheses of Koch's arc theorem and there is an order arc whose supremum is x and whose infimum is y. Let A be the union of a maximal nest of order arcs each of which has x for its supremum. By hypothesis A has an infimum, and by a simple maximality argument that infimum is 0. Thus each element of $S-\{0\}$ is joined to 0 by an arc.

3. **Proof of the main result.** We shall develop a series of lemmas which culminates in a proof of Theorem 1.

LEMMA 1. *If X is a normal space then $C(X)$ is a closed subset of 2^X.*

A proof of Lemma 1 is given in [**5**, p. 139]. We define a relation $\mathcal{I}$ on 2^X (the *inclusion* relation) by $(A, B)\in\mathcal{I}$ if and only if $A\supset B$. Consistent with the notation for partial orders in §2 we also write

$$\mathcal{I}B = \{A \in 2^X : (A, B) \in \mathcal{I}\}$$

and $A\mathcal{I}$ is defined dually. Note that relative to the partial order $\mathcal{I}$, X is a zero for 2^X.

For an alternate proof of Lemma 2, see [**4**, p. 167].

LEMMA 2. *If X is a regular space then $\mathcal{I}$ is a closed subset of $2^X\times 2^X$.*

PROOF. If $(A,B)\in 2^X\times 2^X-\mathcal{I}$ then there exists $b_0\in B-A$ and since X is regular there are disjoint open sets U and V such that $b_0\in U$ and $A\subset V$. Note that $N(A)=\langle V\rangle$ is a neighborhood of A. If A and B have

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

a point in common we set $N(B) = \langle U, X-A, V\rangle$ and otherwise $N(B) = \langle U, X-A\rangle$. In either case $N(B)$ is a neighborhood of B and $N(A)\times N(B)$ contains no member of $\mathfrak{s}$.

LEMMA 3. *If $\mathfrak{N}$ is a nonempty nest which is a closed subset of 2^X or $C(X)$ then* $\mathrm{Cl}(\bigcup\mathfrak{N})\in\mathfrak{N}$.

PROOF. Obviously $\mathrm{Cl}(\bigcup\mathfrak{N})\in 2^X$ and if $\mathfrak{N}\subset C(X)$ then $\mathrm{Cl}(\bigcup\mathfrak{N})\in C(X)$. If $\mathrm{Cl}(\bigcup\mathfrak{N})\in\langle U_1, U_2, \cdots, U_n\rangle$ where the U_i are open subsets of X then $\bigcup\mathfrak{N}$ meets each U_i and hence there exists $N\in\mathfrak{N}$ such that $N\in\langle U_1, U_2, \cdots, U_n\rangle$. Since $\mathfrak{N}$ is closed the lemma follows.

LEMMA 4. *If X is a locally compact, locally connected and connected Hausdorff space and if $Y\in 2^X-\{X\}$ then there exists $Z\in\mathfrak{s}Y-\{Y\}$ such that if $R\in Z\mathfrak{s}\cap\mathfrak{s}Y$ then $Z\mathfrak{s}\cap\mathfrak{s}R$ is a continuum.*

PROOF. Since $Y\neq X$ there exists $y_0\in Y\cap\overline{X-Y}$, and since X is locally connected there exists a continuum N which is a neighborhood of y_0. Hence $Z=N\cup Y$ is a member of $\mathfrak{s}Y-\{Y\}$. Further if $R\in Z\mathfrak{s}\cap\mathfrak{s}Y$ (that is, if $R\in 2^X$ and $Z\supset R\supset Y$) then we can define $\phi: 2^N\to 2^X$ by $\phi(A)=A\cup R$. It is easy to see that ϕ is continuous [**4**, p. 106]; moreover the range of ϕ is precisely $Z\mathfrak{s}\cap\mathfrak{s}R$. Since 2^N is a continuum, so is $Z\mathfrak{s}\cap\mathfrak{s}R$.

LEMMA 5. *Let X be a locally compact, locally connected, connected normal Hausdorff space. If $Y\in C(X)-\{X\}$ then there exists $Z\in(\mathfrak{s}Y-\{Y\})\cap C(X)$ such that if $R\in C(Z)\cap\mathfrak{s}Y$ then $C(R)\cap\mathfrak{s}Y$ is a continuum.*

PROOF. As in the proof of Lemma 4 there exists a continuum N which meets both Y and $X-Y$. In fact, we may assume that N is a locally connected continuum. Let $Z=N\cup Y$ and define $\phi: 2^N\to 2^X$ by $\phi(A)=A\cup Y$. Again ϕ is continuous and since 2^N is compact it follows that $Z\mathfrak{s}\cap\mathfrak{s}Y$ is compact. If R is a connected member of $Z\mathfrak{s}\cap\mathfrak{s}Y$ then by Lemma 2, $R\mathfrak{s}$ is closed in 2^X and hence, by Lemma 1, $C(R)\cap\mathfrak{s}Y$ is compact. Now suppose $C(R)\cap\mathfrak{s}Y$ is not connected. Then it is the union of disjoint closed sets P and Q and we may suppose $R\in P$. Since Q is a compact partially ordered space it contains an $\mathfrak{s}$-minimal element K. (That is, K is a member of Q which is properly contained in no member of Q.) Then there are open subsets $U_1, U_2, \cdots, U_n$ of X such that $K\in\langle U_1, U_2, \cdots, U_n\rangle$ and $\langle\overline{U}_1, \overline{U}_2, \cdots, \overline{U}_n\rangle\cap P$ is empty. Now choose $r\in R-K$; since R is locally compact and connected and since $R-K$ has compact closure, there exists a continuum $B\subset R$ which contains r and meets K. Let $U=U_1\cup U_2\cup\cdots\cup U_n$ and let K_1 be the closure of a component of $B\cap U$ which meets K. It

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

follows that $K\subsetneqq K\cup K_1\subset R$ and $K\cup K_1\in\langle\overline{U}_1, \overline{U}_2, \cdots, \overline{U}_n\rangle$. But then $K\cup K_1$ is a member of Q which contains K properly, and this is a contradiction. This completes the proof that $C(R)\cap\mathcal{I}Y$ is a continuum.

Theorem 1 now follows directly from Theorem 2 and the lemmas. If X is a locally compact Hausdorff space then by Lemma 2, 2^X is a partially ordered space. It has a zero, X, and by Lemma 3 each chain of X has an infimum. If X is connected and locally connected then by Lemma 4 the remaining hypotheses of Theorem 2 are satisfied. If X is also a normal space then Lemma 5 can be invoked instead of Lemma 4 to apply Theorem 2 to $C(X)$.

If X is not locally connected but satisfies all other hypotheses of Theorem 1 then it may happen that neither 2^X nor $C(X)$ is arcwise connected. To see this we recall an example from R. L. Wilder's book [9, p. 102]. In the Cartesian plane let $C=\{(-1, y):0\leqq y\}$, $L=\{(1,y):0\leqq y\}$, let P_n denote the line segment joining $(-n/(n+1),0)$ to $(0, n)$ and let Q_n denote the line segment joining $(0, n)$ to $(1, 0)$. If we set

$$X = C\cup L\cup \bigcup_{n=1}^{\infty}\{P_n\cup Q_n\}$$

then X is a locally compact, connected Hausdorff space which is not locally connected. Now let $U=\{(x, y)\in X:(x+1)(y+1)<1\}$ so that U is an open set which contains C and which meets each of the sets P_n. If α is an arc in 2^X whose endpoints are C and X and if β is the closure of the component of $\alpha\cap\langle U\rangle$ which contains C, then β is an arc whose endpoints are C and some $B_1\in 2^X$ where $B_1\subset\overline{U}$. Consequently $B_1\cap P_n\cap\overline{U}$ is not empty, for some n. In the natural ordering of β from C to B_1 there is a first element which meets $P_n\cap\overline{U}$, say B_0. Let V be an open subset of X which contains $P_n\cap\overline{U}$ but is contained in the complement of C and of each P_k $(k\neq n)$. Then $B_0\in\langle X, V\rangle$ and since $\langle X, V\rangle$ is open in 2^X there exists B between C and B_0 in the arc β with $B\in\langle X, V\rangle$. Since $B\subset U$ it follows that B meets $P_n\cap\overline{U}$, and this contradicts the properties of B_0.

References

1. K. Borsuk and S. Mazurkiewicz, *Sur l'hyperspace d'un continu*, C. R. Soc. Sci. Varsovie **24** (1931), 149–152.

2. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 23–36. MR **3**, 315.

3. R. J. Koch, *Arcs in partially ordered spaces*, Pacific J. Math. **9** (1959), 723–728. MR **21** #7269.

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use

4. K. Kuratowski, *Topology*. Vol. I, rev. ed., Academic Press, New York; PWN, Warsaw, 1966. MR **36** #839; 840.

5. ———, *Topology*. Vol. II, rev. ed., Academic Press, New York; PWN, Warsaw, 1968.

6. M. M. McWaters, *Arcs, semigroups and hyperspaces*, Canad. J. Math. **20** (1968), 1207–1210. MR **37** #6914.

7. E. A. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71** (1951), 152–182. MR **13**, 54.

8. L. E. Ward, Jr., *Concerning Koch's theorem on the existence of arcs*, Pacific J. Math. **15** (1965), 347–355. MR **31** #6206.

9. R. L. Wilder, *Topology of manifolds*, Amer. Math. Soc. Colloq. Publ., vol. 32, Amer. Math. Soc., Providence, R. I., 1949. MR **10**, 614.

University of Oregon, Eugene, Oregon 97403

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use