ARCS IN HYPERSPACES WHICH ARE NOT COMPACT

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Abstract. It has been known for many years that if $X$ is a metrizable continuum then $2^X$ (the space of closed subsets of $X$) and $C(X)$ (the subspace of connected members of $2^X$) are arcwise connected. These results are due to Borsuk and Mazurkiewicz [1] and J. L. Kelley [2], respectively. Quite recently M. M. McWaters [6] extended these theorems to the case of continua which are not necessarily metrizable, using Koch's arc theorem for partially ordered spaces [3], [8]. In this note we prove these results for certain noncompact spaces by means of a simple generalization of Koch's arc theorem.

1. Introduction. Recall that if $X$ is a topological space then $2^X$ denotes the space of nonempty closed subsets of $X$ with the Vietoris topology [7]. That is, if $U_1, U_2, \ldots, U_n$ are subsets of $X$ we write

$$
\langle U_1, U_2, \ldots, U_n \rangle = \left\{ A \in 2^X : A \subseteq \bigcup_{i=1}^{n} \{U_i\} \text{ and } A \cap U_i \neq \emptyset \text{ for each } i = 1, 2, \ldots, n \right\}
$$

and the family of all $\langle U_1, U_2, \ldots, U_n \rangle$ with $U_i$ open is a base for the open sets. The subspace of all closed and connected sets is denoted $C(X)$.

In this note our principal result is a partial extension of the recent theorem of McWaters [6], that if $X$ is an arbitrary continuum, then $2^X$ and $C(X)$ are arcwise connected.

Theorem 1. If $X$ is a locally compact, locally connected and connected Hausdorff space, then $2^X$ is arcwise connected. If, in addition, $X$ is a normal space then $C(X)$ is arcwise connected.

At the end of the paper we give an example which shows that the hypothesis of local connectivity cannot be omitted.

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2. A theorem on partially ordered spaces. A partially ordered space is a topological space $S$ with a partial order $\Gamma$ which is a closed subset of $S \times S$. We treat the symbols $y \leq x$, $y \in \Gamma x$, $x \in \gamma \Gamma$ and $(y, x) \in \Gamma$ as synonyms. Of course $y < x$ means that $y \leq x$ and $y \neq x$. A zero of a partially ordered space $S$ is an element $0 \in S$ such that $0 \Gamma = S$.

McWaters' proof of arcwise connectivity depended on showing that if $X$ is a continuum then $2^X$ and $C(X)$ can be regarded as partially ordered spaces which satisfy the following form of Koch's arc theorem [3], [8]: if $S$ is a compact partially ordered space with zero and if $\Gamma x$ is connected for each $x \in S$, then each nonzero element of $S$ is the supremum of an order arc containing the zero. For the applications of interest to us, a stronger version of Koch's theorem is required.

**Theorem 2.** Let $S$ be a partially ordered space with zero, and suppose that for each nonzero $x \in S$ there exists $y < x$ such that if $y = \neq x$ then $y \Gamma \cap \Gamma t$ is a continuum. If each nonempty chain of $S$ has an infimum then $S$ is arcwise connected.

**Proof.** Let $0$ be the zero of $S$ and let $x \in S - \{0\}$. By hypothesis there exists $y < x$ such that $y \Gamma \cap \Gamma x$ is compact and $y \Gamma \cap \Gamma t$ is connected for each $t \in y \Gamma \cap \Gamma x$. Therefore, $y \Gamma \cap \Gamma x$ satisfies the hypotheses of Koch's arc theorem and there is an order arc whose supremum is $x$ and whose infimum is $y$. Let $A$ be the union of a maximal nest of order arcs each of which has $x$ for its supremum. By hypothesis $A$ has an infimum, and by a simple maximality argument that infimum is $0$. Thus each element of $S - \{0\}$ is joined to $0$ by an arc.

3. Proof of the main result. We shall develop a series of lemmas which culminates in a proof of Theorem 1.

**Lemma 1.** If $X$ is a normal space then $C(X)$ is a closed subset of $2^X$.

A proof of Lemma 1 is given in [5, p. 139]. We define a relation $\mathcal{S}$ on $2^X$ (the inclusion relation) by $(A, B) \in \mathcal{S}$ if and only if $A \supset B$. Consistent with the notation for partial orders in §2 we also write

$$\mathcal{S}B = \{A \in 2^X : (A, B) \in \mathcal{S}\}$$

and $A\mathcal{S}$ is defined dually. Note that relative to the partial order $\mathcal{S}$, $X$ is a zero for $2^X$.

For an alternate proof of Lemma 2, see [4, p. 167].

**Lemma 2.** If $X$ is a regular space then $\mathcal{S}$ is a closed subset of $2^X \times 2^X$.

**Proof.** If $(A, B) \in 2^X \times 2^X - \mathcal{S}$ then there exists $b_0 \in B - A$ and since $X$ is regular there are disjoint open sets $U$ and $V$ such that $b_0 \in U$ and $A \subset V$. Note that $N(A) = \langle V \rangle$ is a neighborhood of $A$. If $A$ and $B$ have
a point in common we set $N(B) = \langle U, X - A, V \rangle$ and otherwise $N(B) = \langle U, X - A \rangle$. In either case $N(B)$ is a neighborhood of $B$ and $N(A) \times N(B)$ contains no member of $\mathcal{G}$.

**Lemma 3.** If $\mathcal{A}$ is a nonempty nest which is a closed subset of $2^X$ or $C(X)$ then $\operatorname{Cl}(U\mathcal{A}) \subseteq \mathcal{A}$.

**Proof.** Obviously $\operatorname{Cl}(U\mathcal{A}) \subseteq 2^X$ and if $\mathcal{A} \subseteq C(X)$ then $\operatorname{Cl}(U\mathcal{A}) \subseteq C(X)$. If $\operatorname{Cl}(U\mathcal{A}) \subseteq \langle U_1, U_2, \ldots, U_n \rangle$ where the $U_i$ are open subsets of $X$ then $U \mathcal{A}$ meets each $U_i$ and hence there exists $N \in \mathcal{A}$ such that $N \subseteq \langle U_1, U_2, \ldots, U_n \rangle$. Since $\mathcal{A}$ is closed the lemma follows.

**Lemma 4.** If $X$ is a locally compact, locally connected and connected Hausdorff space and if $Y \subseteq 2^X - \{X\}$ then there exists $Z \in \mathcal{G} Y - \{Y\}$ such that if $R \subseteq Z \mathcal{G} \cap \mathcal{G} Y$ then $Z \mathcal{G} \cap \mathcal{G} R$ is a continuum.

**Proof.** Since $Y \neq X$ there exists $y_0 \in Y \cap X - Y$, and since $X$ is locally connected there exists a continuum $N$ which is a neighborhood of $y_0$. Hence $Z = N \cup Y$ is a member of $\mathcal{G} Y - \{Y\}$. Further if $R \subseteq Z \mathcal{G} \cap \mathcal{G} Y$ (that is, if $R \subseteq 2^X$ and $Z \supseteq R \supseteq Y$) then we can define $\phi : 2^N \to 2^X$ by $\phi(A) = A \cup R$. It is easy to see that $\phi$ is continuous [4, p. 106]; moreover the range of $\phi$ is precisely $Z \mathcal{G} \cap \mathcal{G} R$. Since $2^N$ is a continuum, so is $Z \mathcal{G} \cap \mathcal{G} R$.

**Lemma 5.** Let $X$ be a locally compact, locally connected, connected normal Hausdorff space. If $Y \subseteq C(X) - \{X\}$ then there exists $Z \in (\mathcal{G} Y - \{Y\}) \cap C(X)$ such that if $R \subseteq C(Z) \cap \mathcal{G} Y$ then $C(R) \cap \mathcal{G} Y$ is a continuum.

**Proof.** As in the proof of Lemma 4 there exists a continuum $N$ which meets both $Y$ and $X - Y$. In fact, we may assume that $N$ is a locally connected continuum. Let $Z = N \cup Y$ and define $\phi : 2^N \to 2^X$ by $\phi(A) = A \cup Y$. Again $\phi$ is continuous and since $2^N$ is compact it follows that $Z \mathcal{G} \cap \mathcal{G} Y$ is compact. If $R$ is a connected member of $Z \mathcal{G} \cap \mathcal{G} Y$ then by Lemma 2, $R \mathcal{G}$ is closed in $2^X$ and hence, by Lemma 1, $C(R) \cap \mathcal{G} Y$ is compact. Now suppose $C(R) \cap \mathcal{G} Y$ is not connected. Then it is the union of disjoint closed sets $P$ and $Q$ and we may suppose $R \subseteq P$. Since $Q$ is a compact partially ordered space it contains an $\mathcal{G}$-minimal element $K$. (That is, $K$ is a member of $Q$ which is properly contained in no member of $Q$.) Then there are open subsets $U_1, U_2, \ldots, U_n$ of $X$ such that $K \subseteq \langle U_1, U_2, \ldots, U_n \rangle$ and $\langle U_1', U_2', \ldots, U_n' \rangle \cap P$ is empty. Now choose $r \in R - K$; since $R$ is locally compact and connected and since $R - K$ has compact closure, there exists a continuum $B \subseteq R$ which contains $r$ and meets $K$. Let $U = U_1' \cup U_2' \cup \cdots \cup U_n'$ and let $K_1$ be the closure of a component of $B \cap U$ which meets $K$. It
follows that $K \subseteq K \cup K_1 \subseteq R$ and $K \cup K_1 \subseteq \langle U_1, U_2, \ldots, U_n \rangle$. But then $K \cup K_1$ is a member of $Q$ which contains $K$ properly, and this is a contradiction. This completes the proof that $C(R) \cap Y$ is a continuum.

Theorem 1 now follows directly from Theorem 2 and the lemmas. If $X$ is a locally compact Hausdorff space then by Lemma 2, $2^X$ is a partially ordered space. It has a zero, $X$, and by Lemma 3 each chain of $X$ has an infimum. If $X$ is connected and locally connected then by Lemma 4 the remaining hypotheses of Theorem 2 are satisfied. If $X$ is also a normal space then Lemma 5 can be invoked instead of Lemma 4 to apply Theorem 2 to $C(X)$.

If $X$ is not locally connected but satisfies all other hypotheses of Theorem 1 then it may happen that neither $2^X$ nor $C(X)$ is arcwise connected. To see this we recall an example from R. L. Wilder's book [9, p. 102]. In the Cartesian plane let $C = \{(-1, y) : 0 \leq y\}$, $L = \{(1, y) : 0 \leq y\}$, let $P_n$ denote the line segment joining $(-n/(n+1), 0)$ to $(0, n)$ and let $Q_n$ denote the line segment joining $(0, n)$ to $(1, 0)$. If we set

$$X = C \cup L \cup \bigcup_{n=1}^{\infty} \{P_n \cup Q_n\}$$

then $X$ is a locally compact, connected Hausdorff space which is not locally connected. Now let $U = \{(x, y) \in X : (x+1)(y+1) < 1\}$ so that $U$ is an open set which contains $C$ and which meets each of the sets $P_n$. If $\alpha$ is an arc in $2^X$ whose endpoints are $C$ and $X$ and if $\beta$ is the closure of the component of $\alpha \cap \langle U \rangle$ which contains $C$, then $\beta$ is an arc whose endpoints are $C$ and some $B_1 \subseteq 2^X$ where $B_1 \subseteq \overline{U}$. Consequently $B_1 \cap P_n \cap \overline{U}$ is not empty, for some $n$. In the natural ordering of $\beta$ from $C$ to $B_1$ there is a first element which meets $P_n \cap \overline{U}$, say $B_0$. Let $V$ be an open subset of $X$ which contains $P_n \cap \overline{U}$ but is contained in the complement of $C$ and of each $P_k$ $(k \neq n)$. Then $B_0 \subseteq \langle X, V \rangle$ and since $\langle X, V \rangle$ is open in $2^X$ there exists $B$ between $C$ and $B_0$ in the arc $\beta$ with $B \subseteq \langle X, V \rangle$. Since $B \subseteq U$ it follows that $B$ meets $P_n \cap \overline{U}$, and this contradicts the properties of $B_0$.

References


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