INVARIANT MEASURES ON LOCALLY COMPACT SEMIGROUPS

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Abstract. The main result of this paper shows that a locally compact abelian semigroup is embeddable as an open subsemigroup of a locally compact abelian group $G$ if and only if the translations $x \mapsto x + y$ are open maps and there exists a nonnegative regular measure $\mu$ on $S$ such that $\mu(U) = \mu(x+U) > 0$ for every open set $U$ and $x$ in $S$.

Our main result is a somewhat stronger statement than the above in that we show that whenever such a measure exists it is the restriction to $S$ of the Haar measure on $G$. This provides a partial answer to a question raised by J. H. Williamson in [5, §5]. We follow the terminology of [5] as regards semigroups and the measure theoretic terminology of [3]. In particular:

A locally compact abelian semigroup $S$ is an abelian semigroup (not necessarily having a unit) which is a locally compact Hausdorff space such that for each $y$ in $S$ the map $x \mapsto x + y$ is continuous. We say that a locally compact abelian semigroup $S$ is embeddable in a locally compact group $G$ if there exists a bicontinuous semigroup monomorphism $\phi$ mapping $S$ into $G$. The following proposition is of independent interest (see [4, Theorem 2.1 and Lemma 1.3]).

Proposition. Let $S$ be a locally compact abelian semigroup. The following conditions on $S$ are equivalent.

1. $S$ is a cancellation semigroup and for each open subset $U$ of $S$, $x + U$ is open for each $x$ in $S$.

2. $S$ is embeddable as an open subsemigroup of a locally compact group $G$.

Proof. It is clear that (2) implies (1). To show that (1) implies (2) let $R$ be the equivalence relation on $S \times S$ defined by $(x, y)R(x_0, y_0)$ if and only if $x + y_0 = y + x_0$. It is well known and easy to show that $G = S \times S/R$ is an abelian group. For $x$ in $S$ let $\phi(x)$ be the equivalence class $\{(x+y, y) : y \in S\}$. The map $\phi : x \mapsto \phi(x)$ is one-one and satisfies $\phi(x+y) = \phi(x) + \phi(y)$. We now define a topology on $G$. For $x$ in $S$ let $\mathcal{B}(x)$ be the neighbourhood filter of $x$. Choose some $x_0$ in $S$ and for
each \( x \) in \( G \) let \( x \cdot \mathcal{B} \) be the filter on \( G \) generated by the filter base \( \{ \phi(U) - \phi(x_0) + x : U \subseteq \mathcal{B}(x_0) \} \). We first show that if \( x \in S \), then \( \phi(x) \cdot \mathcal{B} \) is generated by the filter base \( \{ \phi(U) : U \subseteq \mathcal{B}(x) \} \). Given \( U \subseteq \mathcal{B}(x_0) \), \( U + x \subseteq \mathcal{B}(x + x_0) \) and the continuity of \( \alpha \rightarrow x + \alpha \) at \( x \) means there is a \( V \subseteq \mathcal{B}(x) \) with \( V + x_0 \subseteq U + x \) and so \( \phi(V) \subseteq \phi(U) - \phi(x_0) + \phi(x) \). Conversely given \( V \subseteq \mathcal{B}(x) \), \( V + x_0 \subseteq \mathcal{B}(x + x_0) \) and the continuity of \( \alpha \rightarrow x + \alpha \) at \( x_0 \) means there is a \( U \subseteq \mathcal{B}(x_0) \) such that \( U + x \subseteq V + x_0 \) and so \( \phi(U) - \phi(x_0) + \phi(x) \subseteq \phi(V) \). Therefore \( \phi(x) \cdot \mathcal{B} \) is generated by \( \{ \phi(U) : U \subseteq \mathcal{B}(x) \} \). We now show that there is a unique topology on \( G \) such that for each \( x \) in \( G \), \( x \cdot \mathcal{B} \) is the neighbourhood filter of \( x \). For this it is sufficient by [1, Chapitre 1, §1, No. 1] to show for each \( x \) in \( G \):

(i) if \( V \subseteq x \cdot \mathcal{B} \) then \( x \in V \\
(ii) if \( V \subseteq x \cdot \mathcal{B} \) then there is a \( W \subseteq x \cdot \mathcal{B} \) such that \( y \in W \) implies \( V \subseteq y \cdot \mathcal{B} \).

Clearly (i) is satisfied, so we show (ii). Let \( V \subseteq x \cdot \mathcal{B} \). Then there is an open neighbourhood \( U \) of \( x_0 \) such that \( W = \phi(U) - \phi(x_0) + x \subseteq V \).

If \( y \in W \), there is a \( u \in U \) with \( y = \phi(u) - \phi(x_0) + x \) and there is a \( V' \subseteq \mathcal{B}(x_0) \) such that \( V' + x_0 \subseteq U + x_0 \). Thus

\[
\phi(V') - \phi(x_0) + y = \phi(V') - \phi(x_0) + \phi(u) - \phi(x_0) + x \\
\subseteq \phi(U) - \phi(x_0) + x = W
\]

and therefore \( W \subseteq y \cdot \mathcal{B} \) so that (ii) is satisfied.

It is clear that \( G \) with this topology is a Hausdorff space and that the maps \( x \rightarrow x + y \) are continuous. Moreover for each \( x \) in \( S \), \( \phi(x) \cdot \mathcal{B} \) is generated by \( \{ \phi(U) : U \subseteq \mathcal{B}(x) \} \) so that \( \phi \) is a topological embedding and \( \phi(S) \) is an open subset of \( G \). The continuity and openness of the map \( x \rightarrow x + y \) for each \( y \) in \( G \) together with the local compactness of \( S \) imply that \( G \) is locally compact. Thus \( G \) is a locally compact semigroup which is a group. A theorem of R. Ellis [2, Theorem 2] shows that \( G \) is a locally compact group. This completes the proof.

Theorem. Let \( S \) be a locally compact abelian semigroup and \( \mu \) a nonnegative regular measure on \( S \). Suppose that \( S \) and \( \mu \) satisfy the following condition.

\((*)\) For each open set \( U \), \( x + U \) is open for each \( x \) in \( S \) and \( \mu(x + U) = \mu(U) > 0 \).

Then \( S \) is embeddable as an open subsemigroup in a locally compact abelian group \( G \) and \( \mu \) is the restriction of the Haar measure of \( G \) to \( S \). Conversely if \( S \) is an open subsemigroup of a locally compact abelian group \( G \), and if \( \mu \) is the restriction to \( S \) of the Haar measure of \( G \), then \( S \) is a locally compact abelian semigroup and \( S \) and \( \mu \) satisfy condition \((*)\).
Proof. First suppose $S$ and $\mu$ are given and satisfy (*). We begin by showing that $S$ is a cancellation semigroup. If not there are $x$, $y$, $z$ in $S$ such that $y + x = y + z$ and $x \neq z$. There are open relatively compact neighbourhoods $U$ of $x$ and $V$ of $z$ such that $U \cap V$ is empty. Now

$$\mu((y + U) \cup (y + V)) = \mu(y + (U \cup V)) = \mu(U \cup V)$$

$$= \mu(U) + \mu(V) = \mu(y + U) + \mu(y + V).$$

Since regular measures are by definition finite on compacta it follows that $\mu((y + U) \cap (y + V)) = 0$ which is a contradiction because $(y + U) \cap (y + V)$ is a neighbourhood of $y + x$. Thus $S$ satisfies the hypotheses of the above proposition so $S$ is embeddable as an open subsemigroup of a locally compact abelian group $G$. In the following we identify $S$ with its image in $G$.

Let $\mathcal{K}_S(G)$ be the continuous complex-valued functions on $G$ which are zero outside of $S$ and have compact support. Observe that the invariance property of $\mu$ means that if $f \in \mathcal{K}_S(G)$ then for each $y$ in $S$,

$$\int_S f(x - y) d\mu(x) = \int_S f(x) d\mu.$$

Choose $g \in \mathcal{K}_S(G)$ with $g \geq 0$ and $\int g \ d\lambda = 1$ where $\lambda$ is the Haar measure on $G$. Then using the Fubini Theorem [3, p. 153] we have

$$\int_S f \ d\mu = \int_S f(x - y) \ d\mu(x) \int g(y) \ d\lambda(y)$$

$$= \int_S \int g(y) \ d\lambda(y) \ d\mu(x)$$

$$= \int_S \int f(y) g(x - y) \ d\lambda(y) \ d\mu(x)$$

$$= \int_S g(x - y) \ d\mu(x) \int f(y) \ d\lambda(y)$$

$$= c \int f \ d\lambda$$

where $c = \int_S g \ d\mu$. It follows now that for any Borel subset $E \subset S$, $\mu(E) = c\lambda(E)$ [3, p. 129]. This completes the proof of the first statement.

If $S$ is an open subsemigroup of a locally compact abelian group, then it is clear that $S$ is a locally compact abelian semigroup. More-
over since \( S \) is open the restriction of the Haar measure on \( G \) to \( S \) yields a nonnegative regular measure \( \mu \) such that condition (*) is satisfied.

**References**


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