L² ASYMPTOTES FOR FOURIER TRANSFORMS OF SURFACE-CARRIED MEASURES

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Abstract. W. Littman has shown how to obtain asymptotic approximations for Fourier transforms of surface-carried measures of the form μ(X)dA where dA represents the area measure for the surface as a subset of Euclidean space and μ(X) is a compactly supported C∞ function. Here we extend to the case where μ(X) is an L₁ function.

Let M be the surface in R^{n+1} defined by

\[ X_{n+1} = \phi(X_1, \ldots, X_n) = \phi(X'), \quad \phi \in C^\infty(M'), \]

where M' is an open subset of R^n. For complex valued functions μ on M define

\[ \tilde{\mu}(Y) = \int_{X \in M} e^{iY \cdot X} \mu(X) dA(X), \quad Y \in \mathbb{R}^{n+1}. \]

Define a normal to M at X ∈ M by ν(X) = N(X')/|N(X')| where N(X') = (-grad φ(X'), 1). Let λ₁(X), ..., λ_n(X) be the principle values of the curvature at X ∈ M with signs chosen so that X + [λ_k(X)]^{-1}ν(X) gives the center of curvature for the kth direction. Using Littman [1] one can establish

**Theorem 1.** Assume μ(X) is C∞ and has compact support. Assume M has nowhere zero Gaussian curvature K(X) = Σ_{k=1}^n λ_k(X). Let S^n be the unit sphere in R^{n+1} and assume that the normal ν:M→S^n is a one-to-one map. Define \( a(Y) \) for \( Y \in \{ Y: Y_{n+1} > 0 \} \) as follows: if \( π(Y) \equiv Y/|Y| \) is not in ν(M) put \( a(Y) = 0 \); if \( π(Y) = ν(X), X \in M \), put

\[ a(Y) = e^{iY \cdot X} e^{is/4(2π)^{n/2}} μ(X) \left| Y \right|^{-(n+1)/2} \left| K(X) \right|^{-1/2} \]

where \( s = \sum_{k=1}^n λ_k/|λ_k| \). Then there is a constant B such that \( |\tilde{μ}(Y) - a(Y)| \leq B \left| Y \right|^{-n/2} \left| K(X) \right|^{-1/2} \) for all \( Y \) such that \( |Y| \geq 1 \) and \( Y_{n+1} > 0 \).

A corollary of this theorem is that for \( t > 1 \),

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\[
\|\hat{\mu} - a : t\|_2^2 = \int |\hat{\mu}(Y', t) - a(Y', t)|^2 dY' \\
\leq \int B^2(t^2 + |Y'|^2)^{-(n+1)/2} dY' \\
= t^{-1} \int_0^\infty B^2(1 + r^2)^{-(n+1)/2} n r^{n-1} dr
\]

where \(\omega_n\) is the area of \(S^{n-1}\). For \(\mu\) in \(L^2\) instead of \(C^\infty\) we have

**Theorem 2.** Assume that \(K(X)\) is never zero and that \(\nu : M \rightarrow S^n\) is one-to-one. If

\[
\|\mu\|_2 = \int_M |\mu(X', \phi(X'))|^2 |N(X')|^2 dX' < \infty
\]

then \(\|\hat{\mu} - a : t\| \rightarrow 0\) as \(t \rightarrow \infty\).

**Proof.** We will show

(i) \(\|\chi a : t\|_2^2 = (2\pi)^n \|\mu\|_2^2\) for \(t > 0\);

(ii) \(\|\chi(\hat{\mu} - a) : t\|_2^2 \rightarrow 0\) as \(t \rightarrow \infty\);

(iii) \(\|\hat{\mu} : t\|_2^2 - \|\chi a : t\|_2^2 \rightarrow 0\) as \(t \rightarrow \infty\);

where

\[
\chi(Y) = \begin{cases} 1 & \text{if } \pi(Y) \in \nu(M), \\ 0 & \text{otherwise.} \end{cases}
\]

Since \((1 - \chi)a = 0\), (ii) and (iii) are equivalent to the conclusion of the theorem. By the triangle inequality

\[
\|\chi a : t\| \leq \|\chi(a - \hat{\mu}) : t\| + \|\hat{\mu} : t\|
\]

so that (iii) follows from (i) and (ii).

Using \(dA(X) = |N(X')| dX'\) to rewrite (1) as

\[
\hat{\mu}(Y) = \int_M e^{iy \cdot X'} e^{iY_{n+1} \phi(X')} \mu(X', \phi(X')) |N(X')| dX',
\]

we recognize the right-hand side of (i) as Parseval's equality.

To prove the other half of (i) let

\[
Q_t = \{ Y \in R^{n+1} : Y_{n+1} = t \text{ and } \pi(Y) \in \nu(M) \}
\]

and let \(\pi_t\) be the restriction of \(\pi\) to \(Q_t\). Then
\[
\int_{Y \in G_t} f(Y) dY' = \int_{Y \in S(M)} f(\pi^{-1}(y)) \left| \pi^{-1}(v) \right|/v_{n+1} d\omega(v)
\]

(3) \[
= \int_{X \in M} f(\pi^{-1} \circ \nu(X)) \left| \pi^{-1} \circ \nu(X) \right|^n/\nu_{n+1}(X) |K(X)| dA(X)
\]

\[
= \int_{X \in M} f(tN(X')) |tN(X')|^n |N(X')| |K(X)| dA(X)
\]

where \(d\omega\) denotes surface area on \(S^n\). Since

\[
|a(tN(X'))|^2 = (2\pi)^n |\mu(X)|^2 |tN(X')|^{-n} |K(X)|^{-1}
\]

we see that the left-hand side of (i) follows from (3).

To prove (ii) we use (i) to reduce to the case \(\mu \in C^\infty_c(M)\). Let \(\epsilon > 0\) and choose \(\mu_1 \in C^\infty_c(M)\) such that \(||\mu - \mu_1||_2 < \epsilon(2\pi)^{-n/2}\). Since \(\mu \to a\) is linear, if \(\mu_2 = \mu - \mu_1\) then \(a_2 = a - a_1\). Applying (i) to \(\mu_2\) we have

\[
||x(a - a_1):t|| = \left(2\pi\right)^{n/2} ||\mu - \mu_1||_2 < \epsilon.
\]

Thus

\[
||x(a - a_1):t|| \leq ||x(\mu_1 - \mu_1):t|| + ||x(a - a_1):t|| + ||x(a_1 - a):t||
\]

\[
< \epsilon + ||x(\mu_1 - a_1):t|| + \epsilon.
\]

Since \(\mu_1 \in C^\infty_c(M)\), (2) shows that there exists \(\tau > 0\) such that the last line is less than \(3\epsilon\) for all \(t > \tau\).

For a special case of Theorem 2 with an application see [2].

REFERENCES


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