ABSTRACT MARTINGALES IN BANACH SPACES

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Abstract. The concept of martingale is generalized from probability theory to the setting of Banach spaces. Convergent martingales are characterized. An application to a Radon-Nikodym theorem for vector measures is given.

1. Abstract martingales. Let $X$ be a Banach space and $\{E_t, t \in I\}$ be a uniformly bounded net of continuous linear projections of $X$ into itself satisfying $E_tE_t = E_t, E_t = E_t$ for $t \geq t \in I$. A net $\{x_t, t \in I\} \subset X$ indexed by the same directed set $I$ will be called an abstract martingale and denoted by $\{x_t, E_t, t \in I\}$ if $E_t(x_t) = x_t$ for $t_1, t_2 \in I, t_1 \leq t_2$. Clearly abstract martingales are generalizations of the martingales of probability theory [2], [5], and [8]. On the other hand there are many examples of abstract martingales which do not arise as martingales in the sense of probability theory (see [3, pp. 426–427]). The purpose of this note is to characterize strongly convergent abstract martingales and to indicate briefly some applications including a new Radon-Nikodym theorem for vector valued measures.

Theorem 1. Let $\{x_t, E_t, t \in I\}$ be an abstract martingale in a Banach space $X$. Then $\{x_t, E_t, t \in I\}$ is strongly convergent (i.e. $\lim_{t \to \tau} x_t$ exists strongly in $X$) if and only if there exists a weakly compact set $K \subset X$ such that for each $\varepsilon > 0$ there exists a $t_0 \in I$ such that $t_0, t_0 + \varepsilon \in I$, implying $x_t \in K + \varepsilon U$ ($= \{k + \varepsilon u : k \in K, u \in U\}$) where $U$ is the open unit ball of $X$.

Proof. The necessity is immediate: let $K = \{\lim_{t \to \tau} x_t\}$. Then $\{x_t, t \in I\}$ is eventually in $K + \varepsilon U$ for every choice of $\varepsilon$. To prove the sufficiency of the condition, let $K$ be as in the hypothesis and select $\{\tau_n\} \subset I$ by choosing $\tau_1$ such that $\tau \geq \tau_1$ implies $x_t \in K + U$ and $x_t \in K + (1/n) U$ for $\tau \geq \tau_n$. Now for each $\tau \in I$, choose $z_\tau$ according to the following criteria:

(i) if $\tau \geq \tau_n$ for all $n$, then $x_t \in K$ and $z_\tau$ is taken to be $x_t$;
(ii) if $\tau \geq \tau_n$ and it is not the case that $\tau \geq \tau_{n+1}$, choose $z_\tau \in K$ such that $\|z_\tau - x_t\| < 1/n_0$;
(iii) if there exists no $n$ such that $\tau \geq \tau_n$, choose $z_\tau \in K$ arbitrarily.

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Now consider the net \( \{ z_\tau, \tau \in I \} \subseteq K \). Since \( K \) is weakly compact, there exists a subnet \( \{ z_{\alpha}, \alpha \in A \} \) of \( \{ z_\tau, \tau \in I \} \) converging weakly to some point \( x \in K \). Now let \( f: A \rightarrow I \) be a function which guarantees that \( \{ z_{\alpha}, \alpha \in A \} \) is a subnet of \( \{ z_\tau, \tau \in I \} \) \cite[6, p. 70]{6} and define \( \{ x_{\alpha}, \alpha \in A \} \) by \( x_{\alpha} = x_{f(\alpha)} \). Then \( \{ x_{\alpha}, \alpha \in A \} \) is a subnet of \( \{ x_\tau, \tau \in I \} \) and \( \| x_{\alpha} - z_{\alpha} \| = \| x_{f(\alpha)} - z_{f(\alpha)} \| \). Moreover if \( x^* \in X^* \), the space of bounded linear functionals on \( X \), one has

\[
\lim_{\alpha} \left| x^*(x_{\alpha} - x) \right| \leq \lim_{\alpha} \left| x^*(x_{\alpha} - z_{\alpha}) \right| + \lim_{\alpha} \left| x^*(z_{\alpha} - x) \right|
\]

\[
\leq \| x^* \| \lim_{n} \frac{1}{n} + 0 = 0.
\]

Hence \( \lim_{\alpha} x_{\alpha} = x \) weakly in \( X \). Also since \( \{ x_{\alpha}, \alpha \in A \} \) is a subnet of \( \{ x_\tau, \tau \in I \} \), \( x_\tau = \lim_{\alpha} E_\tau(x_{\alpha}) \) strongly for all \( \tau \in I \). Accordingly if \( \tau \in I \) and \( x^* \in X^* \),

\[
x^*(x_\tau - E_\tau(x)) = x^*(E_\tau(x_\tau - x)) = \lim_{\alpha} x^*E_\tau(x_{\alpha} - x_{\alpha}) = 0,
\]

since \( \lim_{\alpha} x_{\alpha} = x \) weakly and \( x_\tau = \lim_{\alpha} E_\tau(x_{\alpha}) \) strongly. Hence \( x_\tau = E_\tau(x) \) for all \( \tau \in I \).

Finally it will be shown that \( \lim_{\tau} x_\tau = x \) strongly in \( X \). Let \( M = \{ z \in X: E_\tau(z) = z \text{ for some } \tau \in I \} \). The facts that \( I \) is directed and that \( E_\tau E_\tau_1 = E_\tau_1 E_\tau = E_\tau_1 \) for \( \tau \geq \tau_1 \) ensure that \( M \) is a linear manifold in \( X \). But, since \( \lim_{\alpha} x_{\alpha} = x \) weakly and \( \{ x_{\alpha} \} \subseteq M \), \( x \) weakly closed of \( M \) and therefore to the strong closure of the linear manifold \( M \). Now let \( P = \sup_\tau \| E_\tau \| \) and \( \epsilon > 0 \) be given. Choose \( y \in M \) such that \( \| x - y \| < \epsilon/P + 1 \). Selecting \( \tau_0 \in T \) such that \( E_{\tau_0}(y) = y \), one finds that for \( \tau \geq \tau_0 \), \( E_\tau(y) = y \) since \( E_\tau(y) = E_\tau E_{\tau_0}(y) = E_{\tau_0}(y) = y \). Hence for \( \tau \geq \tau_0 \),

\[
\left\| x_\tau - x \right\| = \left\| E_\tau(x) - x \right\| \leq \left\| E_\tau(x) - y \right\| + \left\| y - x \right\|
\]

\[
= \left\| E_\tau(x - y) \right\| + \left\| y - x \right\| < Pe/(P + 1) + \epsilon/(P + 1) = \epsilon.
\]

Q.E.D.

A considerable shortening of the proof of Theorem 1 results in

**Corollary 2.** An abstract martingale is strongly convergent if and only if it is weakly convergent.

Also immediate is

**Corollary 3.** An abstract martingale in a reflexive Banach space is convergent if and only if it is bounded.
2. Applications to martingales and integral representation of vector measures. If $X$ is a reflexive Banach space, and $(\Omega, \Sigma, \mu)$ is a finite measure space, Scalora and Chatterji have shown that a martingale $\{f_n, B_n\}$ in $L^p(\Omega, \Sigma, \mu, X)$ ($=L^p(X)$) converges for $1 < p < \infty$ if and only if $\{f_n, B_n\}$ is bounded [2, Theorem 3]. Since the spaces $L^p(X)$ ($1 < p < \infty$) are reflexive, for reflexive $X$, Corollary 3 contains this result as a special case. In the case $p=1$, Chatterji and Scalora prove that a martingale $\{f_n, B_n\}$ in $L^1(X)$ is convergent if it is bounded and uniformly integrable for reflexive Banach spaces $X$. But, as Chatterji points out [2, p. 145], this assumption guarantees that $\{f_n, B_n\}$ lies in a weakly compact subset of $L^1(X)$. Thus Theorem 1 and its corollary contain the full Chatterji-Scalora theorem on mean convergence of martingales in $L^p(X)$ ($1 \leq p < \infty$). Of course this theorem gives no direct information on almost sure convergence of martingales. On the other hand such information is not to be expected from a theorem of the nature of Theorem 1.

The connection between martingales and derivatives of set functions is well known [8]. The final considerations of this note are devoted to that subject.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space. A partition $\pi = \{E_n\}$ is a finite disjoint collection of sets in $\Sigma$ such that $\bigcup_n E_n = \Omega$. The collection of partitions $P$ becomes a directed set if one defines $\pi \leq \pi'$ if $E \subseteq E'$ implies $E$ is a union of members of $\pi$. Now let $F$ be a $\mu$-continuous countably additive set function defined on $\Sigma$ with values in a Banach space $X$. Define for each partition $\pi = \{E_n\}$ the simple function

$$F_\pi = \sum_r \frac{F(E_n)}{\mu(E_n)} \chi_{E_n}, \quad (0/0) = 0,$$

where $\chi_{E_n}$ is the indicator function of $E_n \subseteq \Sigma$. Then, as Rønnnow [7] has shown for the case $p = 1$ (the same argument holds for all $p \geq 1$) there exists $f \in L^p(\Omega, \Sigma, \mu, X)$ ($1 \leq p < \infty$) such that

$$F(E) = \int_E f d\mu, \quad E \in \Sigma, \text{ (Bochner)}$$

if and only if the net $\{F_\pi, \pi \in P\}$ is a Cauchy net in $L^p(\Omega, \Sigma, \mu, X)$. Now the projections $E_\pi$ defined on $L^p(\Omega, \Sigma, \mu, X)$ for each partition $\pi = \{E_n\}$ by

$$E_\pi(f) = \sum_r \frac{\int_{E_n} f d\mu}{\mu(E_n)} \chi_{E_n}$$
for \( f \in L^p(\Omega, \Sigma, \mu, X) \) are contractions satisfying \( E_\pi E_{\pi_1} = E_{\pi_1} E_\pi = E_\pi \) if \( \pi \succeq \pi_1 \). Now, evidently if \( F \) is as above, then \( \{ F_\pi, E_\pi, \pi \in P \} \) is an abstract martingale in \( L^p(\Omega, \Sigma, \mu, X) \), combining these facts with Theorem 1 results in the following general Radon-Nikodym theorem.

**Theorem 4.** Let \((\Omega, \Sigma, \mu)\) be a finite measure space and \(X\) be a Banach space. Let \(F\) be a \(\mu\)-continuous countably additive \(X\) valued set function defined on \(\Sigma\). Then there exists \( f \in L^p(\Omega, \Sigma, \mu, X) \) \((1 \leq p < \infty)\) such that

\[
F(E) = \int_E fd\mu, \quad E \in \Sigma,
\]

if and only if there exists a weakly compact set \(K \subset L^p(\Omega, \Sigma, \mu, X)\) with the property that for each \(\epsilon < 0\) there exists a partition \(\pi_0\) such that \(\pi \succeq \pi_0\) implies \(F\in K + \epsilon U\) where \(U\) is the open unit ball of \(L^p(\Omega, \Sigma, \mu, X)\).

**References**


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