ABSTRACT MARTINGALES IN BANACH SPACES

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Abstract. The concept of martingale is generalized from probability theory to the setting of Banach spaces. Convergent martingales are characterized. An application to a Radon-Nikodym theorem for vector measures is given.

1. Abstract martingales. Let \( X \) be a Banach space and \( \{ E_\tau, \tau \in I \} \) be a uniformly bounded net of continuous linear projections of \( X \) into itself satisfying \( E_\tau E_\sigma = E_\sigma E_\tau = E_\tau \) for \( \tau \geq \tau_1 \in I \). A net \( \{ x_\tau, \tau \in I \} \subset X \) indexed by the same directed set \( I \) will be called an abstract martingale and denoted by \( \{ x_\tau, E_\tau, \tau \in I \} \) if \( E_{\tau_1}(x_{\tau_2}) = x_{\tau_1} \) for \( \tau_1, \tau_2 \in I, \tau_1 \leq \tau_2 \). Clearly abstract martingales are generalizations of the martingales of probability theory [2], [5], and [8]. On the other hand there are many examples of abstract martingales which do not arise as martingales in the sense of probability theory (see [3, pp. 426–427]). The purpose of this note is to characterize strongly convergent abstract martingales and to indicate briefly some applications including a new Radon-Nikodym theorem for vector valued measures.

Theorem 1. Let \( \{ x_\tau, E_\tau, \tau \in I \} \) be an abstract martingale in a Banach space \( X \). Then \( \{ x_\tau, E_\tau, \tau \in I \} \) is strongly convergent (i.e. \( \lim x_\tau \) exists strongly in \( X \)) if and only if there exists a weakly compact set \( K \subset X \) such that for each \( \varepsilon > 0 \) there exists a \( \tau_0 \in I \) such that \( E_\tau x_\tau \in K + \varepsilon U \) (where \( U \) is the open unit ball of \( X \)).

Proof. The necessity is immediate: let \( K = \{ \lim x_\tau \} \). Then \( \{ x_\tau, \tau \in I \} \) is eventually in \( K + \varepsilon U \) for every choice of \( \varepsilon \). To prove the sufficiency of the condition, let \( K \) be as in the hypothesis and select \( \{ \tau_n \} \subset I \) by choosing \( \tau_n \) such that \( \tau \geq \tau_n \) implies \( x_\tau \in K + U \) and \( \tau_n \geq \tau_{n-1} \) such that \( x_\tau \in K + (1/n)U \) for \( \tau \geq \tau_n \). Now for each \( \tau \in I \), choose \( z_\tau \) according to the following criteria:

(i) if \( \tau \geq \tau_n \) for all \( n \), then \( x_\tau \in K \) and \( z_\tau \) is taken to be \( x_\tau \);

(ii) if \( \tau \geq \tau_{n_0} \) and it is not the case that \( \tau \geq \tau_{n_0+1} \), choose \( z_\tau \in K \) such that \( \| z_\tau - x_\tau \| < 1/n_0 \);

(iii) if there exists no \( n \) such that \( \tau \geq \tau_n \), choose \( z_\tau \in K \) arbitrarily.

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Now consider the net \( \{z_\tau, \tau \in I\} \subseteq K \). Since \( K \) is weakly compact, there exists a subnet \( \{z_\alpha, \alpha \in A\} \) of \( \{z_\tau, \tau \in I\} \) converging weakly to some point \( x \in K \). Now let \( f: A \to I \) be a function which guarantees that \( \{z_\alpha, \alpha \in A\} \) is a subnet of \( \{z_\tau, \tau \in I\} \) \cite[6, p. 70]{6} and define \( \{x_\alpha, \alpha \in A\} \) by \( x_\alpha = x_{f(\alpha)} \). Then \( \{x_\alpha, \alpha \in A\} \) is a subnet of \( \{x_\tau, \tau \in I\} \) and \( \|x_\alpha - z_\alpha\| = \|x_{f(\alpha)} - z_{f(\alpha)}\| \). Moreover if \( x^* \in X^* \), the space of bounded linear functionals on \( X \), one has

\[
\lim_{\alpha} |x^*(x_\alpha - x)| \leq \lim_{\alpha} |x^*(x_\alpha - z_\alpha)| + \lim_{\alpha} |x^*(z_\alpha - x)|
\]

\[
\leq \|x^*\| \lim_{n} \frac{1}{n} + 0 = 0.
\]

Hence \( \lim_\alpha x_\alpha = x \) weakly in \( X \). Also since \( \{x_\alpha, \alpha \in A\} \) is a subnet of \( \{x_\tau, \tau \in I\} \), \( x_\tau = \lim_\alpha E_\tau(x_\alpha) \) strongly for all \( \tau \in I \). Accordingly if \( \tau \in I \) and \( x^* \in X^* \),

\[
x^*(x_\tau - E_\tau(x)) = x^*(E_\tau(x_\tau - x)) = \lim_\alpha x^*E_\tau(x_\alpha - x_\alpha) = 0,
\]

since \( \lim_\alpha x_\alpha = x \) weakly and \( x_\tau = \lim_\alpha E_\tau(x_\alpha) \) strongly. Hence \( x_\tau = E_\tau(x) \) for all \( \tau \in I \).

Finally it will be shown that \( \lim_\tau x_\tau = x \) strongly in \( X \). Let \( M = \{z \in X: E_\tau(z) = z \text{ for some } \tau \in I\} \). The facts that \( I \) is directed and that \( E_\tau E_\tau = E_\tau, E_\tau = E_\tau \) for \( \tau \geq \tau_1 \) ensure that \( M \) is a linear manifold in \( X \). But, since \( \lim_\alpha x_\alpha = x \) weakly and \( \{x_\alpha\} \subseteq M, x \in \text{weak closure of } M \) and therefore to the strong closure of the linear manifold \( M \).

Now let \( P = \sup_\tau \|E_\tau\| \) and \( \epsilon > 0 \) be given. Choose \( y \in M \) such that \( \|x - y\| < \epsilon/P + 1 \). Selecting \( \tau_0 \in T \) such that \( E_{\tau_0}(y) = y \), one finds that for \( \tau \geq \tau_0 \), \( E_\tau(y) = y \) since \( E_{\tau}(y) = E_{\tau_0}E_{\tau}(y) = E_{\tau_0}(y) = y \). Hence for \( \tau \geq \tau_0 \),

\[
\|x_\tau - x\| = \|E_\tau(x) - x\| \leq \|E_\tau(x) - y\| + \|y - x\|
\]

\[
= \|E_\tau(x - y)\| + \|y - x\| < Pe/(P + 1) + \epsilon/(P + 1) = \epsilon.
\]

Q.E.D.

A considerable shortening of the proof of Theorem 1 results in

**Corollary 2.** An abstract martingale is strongly convergent if and only if it is weakly convergent.

Also immediate is

**Corollary 3.** An abstract martingale in a reflexive Banach space is convergent if and only if it is bounded.
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2. Applications to martingales and integral representation of vector measures. If $X$ is a reflexive Banach space, and $(\Omega, \Sigma, \mu)$ is a finite measure space, Scalora and Chatterji have shown that a martingale $\{f_n, B_n\}$ in $L^p(\Omega, \Sigma, \mu, X) (= L^p(X))$ converges for $1 < p < \infty$ if and only if $\{f_n, B_n\}$ is bounded [2, Theorem 3]. Since the spaces $L^p(X) (1 < p < \infty)$ are reflexive, for reflexive $X$, Corollary 3 contains this result as a special case. In the case $p = 1$, Chatterji and Scalora prove that a martingale $\{f_n, B_n\}$ in $L^1(X)$ is convergent if it is bounded and uniformly integrable for reflexive Banach spaces $X$. But, as Chatterji points out [2, p. 145], this assumption guarantees that $\{f_n, B_n\}$ lies in a weakly compact subset of $L^1(X)$. Thus Theorem 1 and its corollary contain the full Chatterji-Scalora theorem on mean convergence of martingales in $L^p(X) (1 \leq p < \infty)$. Of course this theorem gives no direct information on almost sure convergence of martingales. On the other hand such information is not to be expected from a theorem of the nature of Theorem 1.

The connection between martingales and derivatives of set functions is well known [8]. The final considerations of this note are devoted to that subject.

Let $(\Omega, \Sigma, \mu)$ be a finite measure space. A partition $\pi = \{E_n\}$ is a finite disjoint collection of sets in $\Sigma$ such that $\bigcup E_n = \Omega$. The collection of partitions $P$ becomes a directed set if one defines $\pi_1 \leq \pi_2$ if $E \subseteq E_1$ implies $E$ is a union of members of $\pi_2$. Now let $F$ be a $\mu$-continuous countably additive set function defined on $\Sigma$ with values in a Banach space $X$. Define for each partition $\pi = \{E_n\}$ the simple function

$$F_\pi = \sum F(E_n) \frac{\mu(E_n)}{\mu(E_n)} \chi_{E_n}, \quad (0/0) = 0,$$

where $\chi_{E_n}$ is the indicator function of $E_n \subseteq \Sigma$. Then, as Rønnnow [7] has shown for the case $p = 1$ (the same argument holds for all $p \geq 1$) there exists $f \in L^p(\Omega, \Sigma, \mu, X) (1 \leq p < \infty)$ such that

$$F(E) = \int_E f d\mu, \quad E \subseteq \Sigma, \text{ (Bochner)}$$

if and only if the net $\{F_\pi, \pi \in P\}$ is a Cauchy net in $L^p(\Omega, \Sigma, \mu, X)$. Now the projections $E_\pi$ defined on $L^p(\Omega, \Sigma, \mu, X)$ for each partition $\pi = \{E_n\}$ by

$$E_\pi(f) = \sum \frac{\int_{E_n} f d\mu}{\mu(E_n)} \chi_{E_n}$$
for $f \in L^p(\Omega, \Sigma, \mu, X)$ are contractions satisfying $E_{\pi}E_{\pi_1} = E_{\pi_1}E_{\pi} = E_{\pi_1}$ if $\pi \geq \pi_1$. Now, evidently if $F$ is as above, then $\{F_\pi, E_{\pi}, \pi \in P\}$ is an abstract martingale in $L^p(\Omega, \Sigma, \mu, X)$, combining these facts with Theorem 1 results in the following general Radon-Nikodym theorem.

**Theorem 4.** Let $(\Omega, \Sigma, \mu)$ be a finite measure space and $X$ be a Banach space. Let $F$ be a $\mu$-continuous countably additive $X$ valued set function defined on $\Sigma$. Then there exists $f \in L^p(\Omega, \Sigma, \mu, X)$ ($1 \leq p < \infty$) such that

$$F(E) = \int_E f \, d\mu, \quad E \in \Sigma,$$

if and only if there exists a weakly compact set $K \subset L^p(\Omega, \Sigma, \mu, X)$ with the property that for each $\epsilon < 0$ there exists a partition $\pi_0$ such that $\pi \geq \pi_0$ implies $F_\pi \subset K + \epsilon U$ where $U$ is the open unit ball of $L^p(\Omega, \Sigma, \mu, X)$.

**References**