THE POWER PROBLEM FOR GROUPS WITH ONE DEFINING RELATOR

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Abstract. It is proved that if $G$ is a group with one defining relator, then the generalized word problem is solvable for every cyclic subgroup of $G$. This result enables the solution of the word problem for groups with one defining relator to be extended to a wider class of groups.

1. Introduction. Let $G$ be a group given by a presentation $\langle S; D \rangle$, and let $H$ be a subgroup of $G$, generated by a set $A$ of words in the elements of $S$. The generalized word problem (GWP) for $H$ in $G$ is the algorithmic problem of deciding whether or not an arbitrary word $U \in G$ is an element of $H$. If the GWP is solvable for every cyclic subgroup of $G$, then $G$ is said to have solvable power problem. The object of this note is to prove the following

Theorem. Let $G$ have presentation $\langle t, b, c, \ldots ; R(t, b, c, \ldots) \rangle$. Then the power problem is solvable for $G$.

The order problem for a group $G = \langle S; D \rangle$ is the algorithmic problem of deciding the order of an arbitrary word $U \in G$. A solution to this problem for groups with a single defining relator is given by the results of §4.4 of [2]. It follows that Theorem 6 of [3] (see also Theorem 5 of [1]) can be applied to extend the solution of the word problem for groups with a single defining relator to a wider class of groups. As a simple example of this, we have the following generalization of Corollary 4.14.1 of [2].

Corollary. Let $G_1$ and $G_2$ have presentations

$\langle a_1, \ldots, a_m; R(a_1, \ldots, a_m) \rangle$ and $\langle b_1, \ldots, b_n; S(b_1, \ldots, b_n) \rangle$

respectively, and let $U(a_1, \ldots, a_m), V(b_1, \ldots, b_n)$ be elements of $G_1, G_2$ respectively, such that the orders of these elements are equal. Then the order problem and the power problem are solvable for the group

$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n; R, S, U = V \rangle$.

2. Proof of the theorem. We shall make use of the following generalization of Lemma 2 of [1]. The proof requires only trivial
modification of that of Theorem 5 of [1], and so is omitted.

**Lemma.** Let the groups $G_0$ and $G_1$ have presentations

$$\langle a_1, a_2, \ldots, b_1, b_2, \ldots; R_1, R_2, \ldots \rangle$$

and

$$\langle a_1, a_2, \ldots, c_1, c_2, \ldots; S_1, S_2, \ldots \rangle,$$

respectively, and suppose that the following conditions are satisfied:

(a) The power problem is solvable for $G_0$ and $G_1$.

(b) The subgroups $H_i$ of $G_i$ ($i = 0, 1$) generated by the corresponding elements $a_1, a_2, \ldots$ are isomorphic under the identity mapping.

(c) The GWP for $H_i$ in $G_i$ ($i = 0, 1$) is solvable.

Then the power problem is solvable for

$$G = \langle a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots; R_1, R_2, \ldots, S_1, S_2, \ldots \rangle,$$

the free product of $G_0$ and $G_1$ amalgamating $H_0$ with $H_1$.

We prove the theorem by induction on the length of the relator $R$. The method of proof is the one used repeatedly in §4.4 of [2], so we have omitted many of the details.

We can suppose that $R$ as written is cyclically reduced. If $R$ involves only one generator, it is easy to see that the result holds. Thus we assume that $R$ involves at least two generators, say $t$ and $b$, and that the result holds for all groups with one defining relator of length less than that of $R$.

**Case 1.** $R$ has zero exponent sum on some generator; say $\sigma_i(R) = 0$.

Let $U$ and $V$ be elements of $G$. We show that we can decide whether or not there exists an integer $n$ such that $U = V^n$. Using the solution of the order problem for $G$, it is easy to dispose of the case when either $U$ or $V$ has finite order; thus we assume that $U$ and $V$ have infinite order.

Now if $U = V^n$ for some integer $n$ (which must be nonzero), then $UV^{-n} \subseteq N$, the normal subgroup of $G$ generated by $b, c, \ldots$, and so $\sigma_i(UV^{-n}) = 0$. Thus, putting $\lambda = \sigma_i(U)$ and $\eta = \sigma_i(V)$, we must have $\lambda - n\eta = 0$.

Suppose that $\lambda \neq 0$. Then, if $U = V^n$, we have $\eta$ divides $\lambda$ and $n = \lambda / \eta$. Thus in this case there is at most one value of $n$ to test.

Thus we can assume that $\lambda = 0$. If $\eta \neq 0$, then, since $n \neq 0$, we cannot have $\lambda - n\eta = 0$. Thus we can assume also that $\eta = 0$. In other words, we can assume that both $U$ and $V$ are elements of $N$.

We now show that $N$ has solvable power problem. We have
$N = \langle \cdots, b_{-1}, b_0, b_1, \cdots, c_{-1}, c_0, c_1, \cdots, P_{-1}, P_0, P_1, \cdots \rangle,$

where, for $k = 0, \pm 1, \pm 2, \cdots$, $b_k$, $c_k$, $\cdots$ denote the elements $t^k b t^{-k}, t^k c t^{-k}, \cdots$ respectively, and $P_k$ is the element $t^k R t^{-k}$ rewritten in terms of these generators.

Now the subgroup $N_i$ of $N$ generated by

$$\cdots, c_{-1}, c_0, c_1, \cdots, b_{\mu+i}, \cdots, b_{M+i}, \cdots, P_i,$$

where $\mu$ is the minimum subscript on $b$ involved in $P_0$, and $M$ is the maximum subscript on $b$ involved in $P_0$, has presentation

$$N_i = \langle \cdots, c_{-1}, c_0, c_1, \cdots, b_{\mu+i}, \cdots, b_{M+i}; P_i \rangle.$$

Thus $N_i$ is a group with one defining relator $P_i$. Moreover, the length of $P_i$ is less than that of $R$, so that, by the inductive hypothesis, the power problem is solvable for $N_i$.

Now, as in the proof of Theorem 4.14 of [2], we can describe $N$ as the union of a chain of groups

$$Q_1 = N_0 \subseteq Q_2 \subseteq \cdots \subseteq Q_s \subseteq Q_{s+1} \subseteq \cdots.$$

We prove, by induction on $s$, that each $Q_s$ has solvable power problem; it will then follow that $N$ has solvable power problem.

We suppose that the power problem is solvable for $Q_s$. Now $Q_{s+1}$ is the free product of $Q_s$ and some $N_p$, with the subgroup $K$ of $N_p$ generated by all the generators of $N_p$ except some $b_k$ amalgamated under the identity mapping. Denote this set of generators by $A$. Then $A$ is a subset of the generators of some $N_q$ whose generators are among the generators of $Q_s$. Now the GWP for $K$ in $N_p$ except some $b_k$ amalgamated under the identity mapping. Denote this set of generators by $A$. Then $A$ is a subset of the generators of some $N_q$ whose generators are among the generators of $Q_s$. Now the GWP for $K$ in $N_p$ is solvable, as is the GWP for $K$ in $N_q$, by Theorem 4.14 of [2]; moreover, in the proof of that theorem, it is shown that if the generators of $N_q$ are among the generators of $Q_s$, then the GWP for $N_q$ in $Q_s$ is solvable. It follows that the GWP for $K$ in $Q_s$ is solvable. Thus we can apply the lemma, to deduce that the power problem is solvable for $Q_{s+1}$. Hence the power problem is solvable for each $Q_s$, and so is also solvable for $N$.

Case 2. All the generators in $R$ have nonzero exponent sums.

Put $\alpha = \sigma_1(R)$ and $\beta = \sigma_0(R)$. Then $G$ is (effectively) embedded in the group $G_1$ with presentation

$$G_1 = \langle x, y, c, \cdots ; R(y x^{-\beta}, x^\alpha, c, \cdots) \rangle.$$

Thus the power problem is solvable for $G$ if it is solvable for $G_1$. But the exponent sum of $x$ in $R(y x^{-\beta}, x^\alpha, c, \cdots)$ is zero, and when this relator is rewritten in terms of the (usual) generators of the normal
subgroup of $G_1$ generated by $y, c, \cdots$, the relator obtained has length less than that of $R(t, b, c, \cdots)$. Thus, using the same argument as in Case 1, we see that the power problem is solvable for $G_1$. This proves the theorem.

REFERENCES


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