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A UNIQUENESS THEOREM FOR CERTAIN TWO-POINT BOUNDARY VALUE PROBLEMS: A CORRECTION[1]

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ABSTRACT. The boundary value problem $x''=f(t, x, x')$, $x(a)=A$, $x(b)=B$ is shown to have at most one solution on the interval $[a, b]$. The function $f(t, y, z)$ is such that $f(t, y_1, z_1)-f(t, y_2, z_2)>g(t, y_1-y_2, z_1-z_2)$ where initial value problem solutions of $z''=g(t, z, z')$ have a minimum interval of disconjugacy.

The boundary value problem to be considered is

$$\text{BVP(1)} \qquad x'' = f(t, x, x'), \qquad x(a) = A, \quad x(b) = B,$$

where $f(t, y, z)$ is defined on the set

$$T \equiv \{(t, y, z): a \leqq t \leqq b;\ |y| + |z| < \infty;\ a, b \text{ finite}\}.$$

In a recent note J. S. W. Wong [2] stated the following:

THEOREM 1. *Assume that $f(t, y, z)$ and $g(t, y, z)$ are continuous on T. Assume further that*

(1) $f(t, y_1, z_1) - f(t, y_2, z_2) > g(t, y_1 - y_2, z_1 - z_2)$ *for all (t, y_1, z_1) and (t, y_2, z_2) in T such that $y_1>y_2$,*

(2) *the initial value problem*

$$\text{IVP(2)} \qquad z'' = g(t, z, z'), \qquad z(c) = 0, \quad z'(c) = C,$$

for $c \geqq a$ and $C \in R$ has a solution defined for all $t \geqq c$,

(3) *there exists an $h>0$ such that no nontrivial solution $z(t)$ of* IVP(2) *is such that*

$$z(c) = 0 = z(d) \qquad \text{for } 0 < d - c < h,$$

(4) *$g(t, y, z)$ is nondecreasing in y for fixed t and z.*

Then, if $0<b-a \leqq h$, BVP(1) *has at most one solution.*

There appears to be a mistake in the inequalities used by Wong to prove the existence of a function, $\psi(t)$, positive on some interval to

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the right of c. However, the theorem is true and follows immediately from the following well-known result (see, for example, [1, p. 427]).

THEOREM 2. *Assume that $f(t, y, z)$ is continuous on T and that f is strictly increasing in y for fixed t and z. Then* BVP(1) *has at most one solution.*

That Theorem 1 is a corollary of Theorem 2 is clear from condition (1) of Theorem 1 and the following observation.

LEMMA 3. *Assume that $g(t, y, z)$ is continuous on T and is nondecreasing in y for fixed t and z. Assume that there exists an $h>0$ such that no nontrivial solution $z(t)$ of* IVP(2) *is such that $z(c)=0=z(d)$ for $0<d-c<h$. Then $g(t, 0, 0)=0$ for all $t\in[a, b]$.*

PROOF. Given the interval $[a, b]$ and the function $g(t, y, z)$ continuous on T, it follows from standard existence theorems that there exists a $0<\delta_1\leqq b-a$ such that the boundary value problem

$$\text{BVP(3)} \qquad z'' = g(t, z, z'), \qquad z(a_1) = 0, \quad z(b_1) = 0,$$

has a solution on any $[a_1, b_1]\subset[a, b]$ if $0<b_1-a_1\leqq\delta_1$. Let $\delta=\min(\delta_1, h/2)$ and partition the interval $[a, b]$ by the points $a=u_0<u_1<\cdots<u_n=b$ such that for $i\in\{1, 2, \cdots, n\}$, $u_i=\min(u_{i-1}+\delta, b)$. Now with $a_1=u_{i-1}$ and $b_1=u_i$ for $i\in\{1, 2, \cdots, n\}$, BVP(3) has a solution $z(t)$. However $z(t)=0$ for all $t\in[u_{i-1}, u_i]$. Hence the result follows.

We remark that condition (2) is superfluous as is the restriction that $0<b-a\leqq h$. In addition the closing analysis given by Wong as regards the example $g(t, z, z')=-z$ and the interval of uniqueness $b-a\leqq h$ is also invalid.

REFERENCES

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