

## SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

### SHORT PROOF OF SOBCZYK'S THEOREM

WILLIAM A. VEECH<sup>1</sup>

**ABSTRACT.** A new proof is given of Sobczyk's Theorem, which asserts that  $c_0$  is complemented in any separable Banach space which contains it (isomorphically) as a closed subspace.

Sobczyk's Theorem says that for every separable Banach space  $X$  and isometric copy  $Y$  of  $c_0$  contained in  $X$ , there is a projection of norm at most 2 from  $X$  to  $Y$  (and 2 is best possible). This is a consequence of setting  $T_0 = \text{identity}$  in the following theorem, whose equivalence to Sobczyk's Theorem seems to be part of the folklore.

**THEOREM.** *Let  $X$  be a separable Banach space and  $Y$  a closed subspace of  $X$ . If  $T_0: Y \rightarrow c_0$  is a linear operator of norm  $\lambda$ , there exists an extension  $T: X \rightarrow c_0$  of norm at most  $2\lambda$ .*

**PROOF.**  $S_\lambda$  denotes the closed ball of radius  $\lambda$  in the dual of  $X$ . In the weak\* topology  $S_\lambda$  is compact, and because  $X$  is separable, metrizable. Let  $d(\cdot, \cdot)$  be a compatible metric. By the Hahn-Banach theorem there exists a sequence  $\phi_1, \phi_2, \dots$  in  $S_\lambda$  such that the  $n$ th coordinate of  $T_0 y$ ,  $y \in Y$ , is  $(T_0 y)_n = \phi_n(y)$ . Letting  $K = S_\lambda \cap Y^\perp$ , it is clear every cluster point of  $\{\phi_n\}$  lies in  $K$ . What is the same,  $d(\phi_n, K) \rightarrow 0$ , and if  $\{\psi_n\} \subseteq K$  is a sequence minimizing  $d(\phi_n, \psi)$ ,  $\psi \in K$ , 0 is the only cluster point of  $\{\phi_n - \psi_n\} \subseteq S_{2\lambda}$ . Define  $(Tx)_n = \phi_n(x) - \psi_n(x)$ ,<sup>2</sup>  $x \in X$ .  $T$  maps  $X$  to  $c_0$ , has norm at most  $2\lambda$ , and agrees with  $T_0$  on  $Y$ . The proof is complete.

Haskell Rosenthal has pointed out to us that our argument uses only the weak\* metrizable of the closure of  $\{\phi_n\}$  in  $S_\lambda$  and there-

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<sup>2</sup> ADDED IN PROOF. Since this was written we have learned this device has been used earlier by Köthe for another result, also equivalent to Sobczyk's Theorem [3], [4].

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fore is valid for any “weakly compactly generated” Banach space  $X$  (those Banach spaces  $X$  having weakly compact total subsets).

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RICE UNIVERSITY, HOUSTON, TEXAS 77001