THE NORM OF A HERMITIAN ELEMENT IN A BANACH ALGEBRA

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Abstract. We prove that the norm of a hermitian element in a Banach algebra is equal to the spectral radius of the element.

An element \( h \) in a complex Banach algebra with identity (of norm 1) is said to be hermitian if \( \|\exp i\alpha h\| = 1 \) for all real \( \alpha \) [6], [3, Definition 5.1]. I. Vidav uses a Phragmén-Lindelöf theorem to show that the numerical radius [3, Definition 2.1] of a hermitian element is equal to its spectral radius [6, p. 123, Hilfssatz 3], [3, Theorem 5.10]. We show that the norm of \( h + \beta i \) is equal to the spectral radius of \( h + \beta i \) for \( h \) a hermitian element and \( \beta \) a complex number (Proposition 2). The proof uses a generalisation of Bernstein’s theorem which gives a bound on the derivative of an entire function along the real line. F. F. Bonsall and M. J. Crabb [2] have recently given an elementary proof of our Proposition 2 when \( \beta \) is zero (which is equivalent to \( \beta \) real). In Lemma 5 and Proposition 6 we construct a norm on the algebra of polynomials, in one indeterminate \( x \), which is maximal with respect to the property that \( x \) is hermitian of norm one.

An entire function \( F \) is said to be of order \( R \) if
\[
R = \limsup_{\alpha \to \infty} \frac{\log \log M(\alpha)}{\log \alpha}
\]
where \( M(\alpha) \) denotes \( \sup\{ |F(z)| : |z| \leq \alpha \} \). An entire function of finite order \( R \) is said to be of type \( T \) if
\[
T = \limsup_{\alpha \to \infty} \alpha^{-R} \log M(\alpha).
\]
If the entire function \( F \) is of order less than 1 or \( F \) is of order 1 and type less than or equal to \( T \), we say \( F \) is of exponential type \( T \) [1, p. 8]. G. Lumer and R. S. Phillips [5, p. 685, Theorem 2.3] prove the following lemma when \( x \) is topologically nilpotent. Let \( \nu(x) \) denote the spectral radius of an element \( x \).

1. Lemma. Let \( A \) be a Banach algebra with identity. For each continuous linear functional \( f \) on \( A \) and each \( x \) in \( A \), the entire function \( \lambda \to f(\exp \lambda x) \) is of exponential type \( \nu(x) \).

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Proof. Since \(|f(\exp \lambda x)| \leq \|f\| \cdot \exp |\lambda| \cdot |x|\), we see that the order of \(f(\exp \lambda x)\) is less than or equal to 1. Suppose that the order of \(f(\exp \lambda x)\) is 1. The \(n\)th derivative of \(f(\exp \lambda x)\) at zero is \(f(x^n)\). Thus, by equation 2.2.12 of [1, p. 11], the type of \(f(\exp \lambda x)\) is equal to \(\lim sup_{n \to \infty} \left|f(x^n)\right|^{1/n}\), which is less than or equal to the spectral radius of \(x\). This completes the proof.

Alternatively Lemma 1 may be proved using [3, Theorem 3.8].

For each \(x\) in \(A\) that is not topologically nilpotent there is a continuous linear functional \(f\) on \(A\) with \(\|f\| = f(1) = 1\) such that \(f(\exp \lambda x)\) has order 1 and type \(\nu(x)\). Let \(B\) be a closed commutative subalgebra of \(A\) containing \(x\) and 1, and let \(\theta\) be a character on \(B\) such that the modulus of \(\theta(x)\) is equal to \(\nu(x)\). By the Hahn-Banach theorem there is an extension \(f\) of \(\theta\) to \(A\) of norm 1. Then \(f(\exp \lambda x) = \exp \lambda \theta(x)\), which is of order 1 and type \(\nu(x)\).

2. Proposition. Let \(A\) be a Banach algebra with identity. Then \(\|h + \beta I\| = \nu(h + \beta I)\) for each hermitian element \(h\) and each complex number \(\beta\).

Proof. Because the sum of two hermitian elements is hermitian and a real multiple of the identity is hermitian [6, p. 122, Hilfssatz 2], [3, Lemma 5.4], we have to prove \(\|h + \beta I\| = \nu(h + \beta I)\) only when \(\beta\) is imaginary. Let \(\gamma\) be a real number, and let \(f\) be a continuous linear functional on \(A\) of norm 1 with \(f(h + i\gamma I) = \|h + i\gamma I\|\). Then, by Lemma 1, \(\lambda \to f(\exp \lambda i\gamma)\) is an entire function of exponential type \(\nu(h)\) whose modulus is bounded by 1 for all real \(\lambda\). We now state a generalization of a theorem of S. Bernstein [4, Theorem 1], [1, Chapter 11]. If \(F\) is an entire function of exponential type \(T\) whose modulus is bounded by 1 for all real \(\lambda\). We now state a generalization of a theorem of S. Bernstein [4, Theorem 1], [1, Chapter 11]. If \(F\) is an entire function of exponential type \(T\) whose modulus is bounded by 1 for all real \(\lambda\), then

\[
|F'(\lambda) - \alpha F(\lambda)| \leq (T^2 + \alpha^2)^{1/2}
\]

for all real \(\lambda\) and \(\alpha\), where \(\cdot\)' denotes differentiation with respect to \(\lambda\). Although the hypotheses of [4, Theorem 1] are not stated in terms of the type of an entire function it is a routine matter to write them in this form so that (1) is a special case of [4, Theorem 1]. Alternatively, when \(T\) is nonzero this inequality may be obtained from inequality 11.4.5 of [1, p. 214] by substituting \(\alpha = -T \tan \omega\) (see also [1, p. 211 and p. 222]).

We apply (1) with \(F(\lambda) = f(\exp \lambda i\gamma)\) and \(\lambda = 0\) obtaining

\[
\|h + i\gamma I\| = |f(h) + i\gamma f(1)| \leq |\nu(h) + i\gamma|
\]

since the derivative of \(f(\exp \lambda i\gamma)\) is \(f(ih \exp \lambda i\gamma)\). Since the spectrum
of \( h \) is contained in the real line \([6, \text{p. 122, Hilfssatz 2}]\) and \( \gamma \) is real,

\[ \nu(h + iy) = |\nu(h) + iy| . \]

Combining (2) and (3) completes the proof.

We shall require the following corollary in Proposition 6.

3. COROLLARY. If \( Q \) is a polynomial, with complex coefficients, whose zeros lie on the imaginary axis, and if \( h \) is a hermitian element, then \( \|Q(h)\| = |Q|\|h\| \).

PROOF. The spectrum of \( h \) is contained in the real line, and so, by Proposition 2, \( \|h\| \) or \( -|h| \) is in \( \sigma(h) \). Thus \( \nu(h-\alpha) = \|h\| - \alpha \) for all imaginary \( \alpha \). Proposition 2 now implies that \( \|h-\alpha\| = \|h\| - \alpha \).

We factorise \( Q(h) \) into linear factors and use this result and the sub-multiplicativity of the norm to obtain \( \|Q(h)\| \leq |Q|\|h\| \). As all the zeros of \( Q \) lie on the imaginary axis, \( |Q(h)| = |Q(-h)| \). This and the result that \( \|h\| \) or \( -|h| \) is in \( \sigma(h) \) imply that \( |Q(h)| \leq \nu(Q(h)) \) \( \leq \|Q(h)\| \), which completes the proof.

Alternatively Corollary 3 may be proved directly from Lemma 1 by using Theorems 11.7.7, 7.8.3, and 11.7.2 of [1].

4. DEFINITION. Let \( C(x) \) be the algebra of all polynomials in \( x \) with complex coefficients, and let \( L \) be the set of all constants, and all polynomials whose zeros lie on the imaginary axis in the complex plane. Then every polynomial \( P \) in \( C(x) \) is the sum of a finite number of polynomials in \( L \). Let \( \alpha \) be positive real number. We define \( \| - \|_0 \) on \( C(x) \) by

\[ \|P\|_0 = \inf \left\{ \sum_j |Q_j(\alpha)| : P = \sum_j Q_j, Q_j \in L \text{ all } j \right\}, \]

and \( \| - \|_\infty \) on \( C(x) \) by

\[ \|P\|_\infty = \sup \{ |P(\lambda)| : -\alpha \leq \lambda \leq \alpha \}. \]

5. LEMMA. Let \( \alpha \) be a positive real number. Then \( \| - \|_0 \) (and \( \| - \|_\infty \)) is an algebra norm on \( C(x) \), \( \| - \|_0 \leq \| - \|_\infty \), and \( x \) is a hermitian element in the completion of \( (C(x), \| - \|_0) \) with spectrum the interval \([-\alpha, \alpha]\).

PROOF. If \( Q \) is in \( L \), then \( \beta \rightarrow |Q(\beta)| \) is a monotonically increasing function of positive real \( \beta \), as may be seen by factorising \( Q \) into linear factors and noting that the zeros of \( Q \) lie on the imaginary axis so that \( \beta \rightarrow |\beta - \gamma| \) is a monotonically increasing function for each zero \( \gamma \) of \( Q \). Let \( P = \sum_j Q_j \) with \( Q_j \) in \( L \), and let \( -\alpha \leq \lambda \leq \alpha \). Then \( \|P(\lambda)\| \leq \sum_j |Q_j(\lambda)| \) since \( \lambda \) is real, since the zeros of \( Q_j \) lie on the imaginary axis, and since \( |\lambda + i\alpha| = |\gamma + i\alpha| \) for all \( \alpha \). There-
fore \(|P(\lambda)| \leq \sum_i |Q_i(\alpha)|\), so that \(||P||_\infty \leq ||P||_0\). If \(||P||_0 = 0\), \(P\) is zero on \([-\alpha, \alpha]\) and so \(P = 0\). An elementary calculation now shows that \(||\cdot||_0\) is an algebra norm on \(C(x)\).

Let \(A\) be the completion of \(C(x)\) in \(||\cdot||_0\). Then, for all real \(t\), \(\exp itx\) is the \(||\cdot||_0\)-limit of \((1 + i/n \cdot tx)^n\) as \(n\) tends to infinity [3, Theorem 3.3]. Now \(||(1 + i/n \cdot tx)^n||_0 \leq |(1 + i/n \cdot tx)^n|\), so that, taking limits as \(n\) tends to infinity, we obtain \(||\exp itx||_0 \leq |\exp itx| = 1\). Therefore \(||\exp itx||_0 = 1\) for all real \(t\), so that \(x\) is a hermitian element in \(A\).

Since \(x\) is hermitian and \(||x||_0 \leq \alpha\), the spectrum of \(x\) in \(A\) is contained in the interval \([-\alpha, \alpha]\). For each \(\lambda\) in \([-\alpha, \alpha]\) the function \(P \mapsto P(\lambda) : C(x) \rightarrow \mathbb{C}\) is a continuous character on \(C(x)\) taking the value \(\lambda\) at \(x\). This shows that the spectrum of \(x\) in \(A\) is \([-\alpha, \alpha]\), and completes the proof.

The norm \(||\cdot||_0\) given above is the maximal norm on \(C(x)\) such that \(x\) is hermitian with \(||x|| = \alpha\).

6. Proposition. Let \(\alpha\) be a positive real number, let \(A\) be a Banach algebra with identity, and let \(h\) be in \(A\). Then \(h\) is hermitian with \(||h|| \leq \alpha\) if, and only if, \(||P(h)|| \leq ||P||_0\) for all \(P\) in \(C(x)\).

Proof. If \(h\) is hermitian with \(||h|| \leq \alpha\), then, for each \(Q\) in \(L\), \(||Q(h)|| = |Q(||h||)\) by Corollary 3 and the monotonicity of \(|Q(\alpha)|\), which we proved in Lemma 5. Thus \(||P(h)|| \leq \sum_j |Q_j(\alpha)|\) for all \(Q_j\) in \(L\) with \(P = \sum_j Q_j\), so that \(||P(h)|| \leq ||P||_0\) for all \(P\) in \(C(x)\).

Conversely, suppose that \(||P(h)|| \leq ||P||_0\) for all \(P\) in \(C(x)\). Then, by [3, Theorem 3.3],

\[||\exp ith|| = \lim_{n \to \infty} ||(1 + i/n \cdot th)^n|| \leq \lim_{n \to \infty} ||(1 + i/n \cdot tx)^n||_0\]

for all real \(t\). This implies that \(||\exp ith|| = 1\) for all real \(t\), and completes the proof since \(||h|| \leq ||x||_0 \leq \alpha\).

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References


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