A THEOREM ON PERFECT MAPS

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1. Introduction. The purpose of this note is to give a short proof of the following theorem, and to indicate some applications.

Theorem 1.1. If $f:X \to Y$ is perfect, and $g:X \to Z$ is continuous with $Z$ Hausdorff, then $(f, g)_*: X \to Y \times Z$ is perfect.

Theorem 1.1 is implicit in the proofs of two results of A. V. Arhangel'skiï [1, Lemmas 1 and 3], and also follows immediately from a result on set-valued maps which is stated by Z. Frolik in [3, Proposition 6 and remark at end of §1]. We prove Theorem 1.1 in §2.

The following is a direct consequence of Theorem 1.1.

Corollary 1.2. If $X$ admits a perfect map into a topological space $Y$, and a continuous one-to-one map into a Hausdorff space $Z$, then $X$ is homeomorphic to a closed subspace of $Y \times Z$.

Corollary 1.2 immediately implies the nontrivial part ($(a) \to (b)$) of the following result, which was essentially obtained by J. Nagata in [4, Theorem 1], and which also follows from J. van der Slot [5, Theorem, p. 21].

Corollary 1.3. If $Y$ is any topological space, then the following properties of a completely regular space $X$ are equivalent.

(a) There exists a perfect map $f:X \to Y$.

(b) $X$ is homeomorphic to a closed subspace of $Y \times Z$ for some compact Hausdorff space $Z$.

(c) $X$ is homeomorphic to a closed subspace of $Y \times Z$ for some compact space $Z$.

In a different direction, the following result of Bourbaki [2, p. 115,

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2 A map $f:X \to Y$ (not necessarily onto) is perfect if $f$ is closed (i.e. $f(A)$ is closed in $Y$ for every closed $A \subseteq X$) and $f^{-1}(y)$ is compact for every $y \in Y$. (Perfect maps are called proper by Bourbaki [2].)

3 We define $(f, g)_*(x) = (f(x), g(x))$.

4 It appears that Arhangel'skiï calls a map $f:X \to Y$ perfect in [1] if the map $f:X \to f(X)$ is perfect in our terminology. Thus Arhangel'skiï does not require $f(X)$ to be closed in $Y$.

5 I am grateful to A. V. Arhangel'skiï for this reference.

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Proposition 5(d)] is also an easy consequence of Theorem 1.1, as our proof in §3 will show.⁶

**Corollary 1.4.** Let \( \alpha : A \to B \) and \( \beta : B \to C \) be continuous, and suppose that \( \beta \circ \alpha \) is perfect and that \( B \) is Hausdorff. Then \( \alpha \) is perfect.

In conclusion, let us observe that the following useful known result follows immediately from Corollary 1.4 (by taking \( \alpha : A \to B \) to be the injection map).

**Corollary 1.5.** If \( \gamma : A \to C \) is perfect, and if \( \gamma \) has a continuous extension \( \beta : B \to C \) for some Hausdorff space \( B \supset A \), then \( A \) is closed in \( B \).

2. **Proof of Theorem 1.1.** Clearly \( (f, g) \) is the composition of the following two maps:

\[
\begin{array}{c}
X \\
\xrightarrow{(i_x, g)}
\end{array}
\begin{array}{c}
X \\
\times Z \\
\xrightarrow{f \times i_Z}
\end{array}
\begin{array}{c}
Y \\
\times Z
\end{array}
\]

Now \( (i_x, g) \) maps \( X \) homeomorphically onto the graph of \( g \), which is closed in \( X \times Z \) because \( Z \) is Hausdorff. Since \( f \times i_Z \) is the product of two perfect maps, it is perfect by [2, p. 114, Proposition 4]. Hence \((f, g)\) is perfect.⁷

3. **Proof of Corollary 1.4.** If \( \gamma = (\alpha, \beta \circ \alpha) \), then \( \gamma : A \to B \times C \) is perfect by Theorem 1.1. Now the projection \( \pi : B \times C \to B \) maps the graph \( G_\beta \) of \( \beta \) homeomorphically onto \( B \). Since \( \gamma(A) \subseteq G_\beta \) and \( \alpha = (\pi | G_\beta) \circ \gamma \), it follows that \( \alpha \) is perfect.

**References**


5. J. van der Slot, Some properties related to compactness, Mathematical Center Tracts 19, Amsterdam, 1966.

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⁶ As a partial converse, Corollary 1.4 implies the slight weakening of Theorem 1.1 which results from assuming that \( Y \) (as well as \( Z \)) is Hausdorff.

⁷ The assumption that \( Z \) is Hausdorff cannot be dropped, or even weakened to \( T_1 \). Example: \( X = Y = \text{interval} \ I \) with usual topology, \( Z = I \) with cofinite topology, \( f = g = i_x \).