A THEOREM ON PERFECT MAPS

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1. Introduction. The purpose of this note is to give a short proof of the following theorem, and to indicate some applications.

Theorem 1.1. If \( f: X \to Y \) is perfect,2 and \( g: X \to Z \) is continuous with \( Z \) Hausdorff, then \( (f, g): X \to Y \times Z \) is perfect.3

Theorem 1.1 is implicit in the proofs of two results of A. V. Arhangel’skiï [1, Lemmas 1 and 3],4 and also follows immediately from a result on set-valued maps which is stated by Z. Frolik in [3, Proposition 6 and remark at end of §1]. We prove Theorem 1.1 in §2.

The following is a direct consequence of Theorem 1.1.

Corollary 1.2. If \( X \) admits a perfect map into a topological space \( Y \), and a continuous one-to-one map into a Hausdorff space \( Z \), then \( X \) is homeomorphic to a closed subspace of \( Y \times Z \).

Corollary 1.2 immediately implies the nontrivial part ((a) \( \to \) (b)) of the following result, which was essentially obtained by J. Nagata in [4, Theorem 1], and which also follows from J. van der Slot [5, Theorem, p. 21].6

Corollary 1.3. If \( Y \) is any topological space, then the following properties of a completely regular space \( X \) are equivalent.

(a) There exists a perfect map \( f: X \to Y \).

(b) \( X \) is homeomorphic to a closed subspace of \( Y \times Z \) for some compact Hausdorff space \( Z \).

(c) \( X \) is homeomorphic to a closed subspace of \( Y \times Z \) for some compact space \( Z \).

In a different direction, the following result of Bourbaki [2, p. 115,
Proposition 5(d)] is also an easy consequence of Theorem 1.1, as our proof in §3 will show.\footnote{As a partial converse, Corollary 1.4 implies the slight weakening of Theorem 1.1 which results from assuming that $Y$ (as well as $Z$) is Hausdorff.}

**Corollary 1.4.** Let $\alpha: A \to B$ and $\beta: B \to C$ be continuous, and suppose that $\beta \circ \alpha$ is perfect and that $B$ is Hausdorff. Then $\alpha$ is perfect.

In conclusion, let us observe that the following useful known result follows immediately from Corollary 1.4 (by taking $\alpha: A \to B$ to be the injection map).

**Corollary 1.5.** If $\gamma: A \to C$ is perfect, and if $\gamma$ has a continuous extension $\beta: B \to C$ for some Hausdorff space $B \supseteq A$, then $A$ is closed in $B$.

2. **Proof of Theorem 1.1.** Clearly $(f, g)$ is the composition of the following two maps:

\[
X \xrightarrow{(i_x, g)} X \times Z \xrightarrow{f \times i_z} Y \times Z.
\]

Now $(i_x, g)$ maps $X$ homeomorphically onto the graph of $g$, which is closed in $X \times Z$ because $Z$ is Hausdorff. Since $f \times i_z$ is the product of two perfect maps, it is perfect by [2, p. 114, Proposition 4]. Hence $(f, g)$ is perfect.\footnote{The assumption that $Z$ is Hausdorff cannot be dropped, or even weakened to $T_1$. Example: $X = Y =$ interval $I$ with usual topology, $Z = I$ with cofinite topology, $f = g = i_x$.}

3. **Proof of Corollary 1.4.** If $\gamma = (\alpha, \beta \circ \alpha)$, then $\gamma: A \to B \times C$ is perfect by Theorem 1.1. Now the projection $\pi: B \times C \to B$ maps the graph $G_\beta$ of $\beta$ homeomorphically onto $B$. Since $\gamma(A) \subseteq G_\beta$ and $\alpha = (\pi | G_\beta) \circ \gamma$, it follows that $\alpha$ is perfect.

**References**


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