INTEGRAL RING EXTENSIONS AND PRIME IDEALS OF INFINITE RANK

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Abstract. An example is constructed showing that for an integral ring extension $R \subseteq T$, and a prime ideal $P$ of $R$ having infinite rank, it can happen that in $T$ each prime ideal lying over $P$ has finite rank.

By the rank (or height) of a prime ideal $P$ in a commutative ring $R$ is meant the maximal length of descending chains of prime ideals of $R$ starting with $P$. Thus $P$ has rank $n$ if there exists a descending chain $P = P_0 \supset P_1 \supset \cdots \supset P_n$, but no such chain of longer length; and $P$ has infinite rank (or rank $\infty$) if there exist arbitrarily long chains of primes descending from $P$. Let $R \subseteq T$ be a pair of commutative rings (having a common identity). One says that the going up property (GU) holds for the pair $R \subseteq T$ if whenever $P \subseteq P_0$ are prime ideals in $R$ and $Q$ is a prime of $T$ such that $Q \cap R = P$, then there exists a prime $Q_0$ in $T$ such that $Q \subseteq Q_0$ and $Q_0 \cap R = P_0$. It is well known that if $T$ is integral over $R$, then GU holds for the pair $R \subseteq T$; and it can be readily seen that if $R \subseteq T$ satisfies GU and $P$ is a prime ideal in $R$ of rank $n$, then there exists in $T$ a prime ideal $Q$ such that $Q$ has rank $\geq n$ and $Q \cap R = P$ [3, Theorem 46, p. 31]. We show, however, that this result cannot be extended to primes of rank $\infty$ even for $R$ an integral domain and $T$ the integral closure of $R$. Of course GU insures that there can be no fixed bound on the ranks of the primes of $T$ lying over a rank $\infty$ prime $P$ of $R$. Thus in our example there must be infinitely many primes of $T$ lying over $P$. In particular, $T$ cannot be a finite $R$-module [1, p. 40].

The idea involved in our construction may be stated as follows.

Lemma. Let $R$ be a quasi-local domain with maximal ideal $P$ and quotient field $K$. Assume that for each positive integer $n$ there exists a valuation ring of $K$ containing $R$ and having rank $n$, but that $R$ is contained in no valuation ring of $K$ having infinite rank. Let $T$ be the integral closure of $R$. If $T$ is a Prüfer domain, then $P$ has infinite rank but each prime ideal of $T$ has finite rank.

Proof. If $Q$ is a prime ideal of $T$, then the localization $T_Q$ is a...
valuation ring and the rank of the prime ideal \( Q \) equals the rank of the valuation ring \( T_Q \). Thus each prime of \( T \) has finite rank and by intersecting chains of primes of \( T \) with \( R \), we see that \( P \) has infinite rank.

**Construction of the example.** Let \( k \) be an arbitrary field and let \( \{ x_i \}_{i=1}^\infty \) be a collection of indeterminates over \( k \). We construct a rank one valuation ring \( V_1 \) on the field \( K = k(x_1, x_2, \ldots) \) such that \( V_1 \) has the form \( k + M_1 \) where \( M_1 \) is the maximal ideal of \( V_1 \). This can be done, for example, by mapping the \( x_i \) onto rationally independent real numbers and then extending this map to a valuation of \( K \) trivial on \( k \). The \( x_i \) having rationally independent values assures that \( k \) maps isomorphically onto the residue field of \( V_1 \) and hence that \( V_1 = k + M_1 \). For each integer \( n \geq 2 \), let \( L_n \) denote the field \( k(\{ x_i \mid i \leq n \text{ or } i \geq 2n \}) \). Thus \( K = L_n(\{ x_{n+1}, \ldots, x_{2n-1} \}) \) and \( x_{n+1}, \ldots, x_{2n-1} \) are algebraically independent over \( L_n \). Consider the valuation ring \( V_1 \cap L_n \). By mapping \( x_{n+1}, \ldots, x_{2n-1} \) onto suitably chosen elements of a suitable totally ordered abelian group containing the value group of \( V_n \), we can obtain a valuation ring \( V_n \) of \( K \) such that:

1. \( V_n \cap L_n = V_1 \cap L_n \).
2. \( V_n \) has rank \( n \).
3. \( V_n \) has the form \( k + M_n \) where \( M_n \) is the maximal ideal of \( V_n \).

See, for example, [1, Proposition 1, p. 161].

Let \( P = \bigcap_{i=1}^\infty M_i \) and let \( R = k + P \). We note that \( R \) is a quasi-local domain with maximal ideal \( P \). For if \( \alpha \) is a nonzero element of \( k \) and \( m \in P \), then \( (\alpha + m)^{-1} = \alpha^{-1} + m' \), where \( m' = -m/\alpha(\alpha + m) \in M_i \) for each \( i \), so \( m' \in P \). Let \( T \) be the integral closure of \( R \).

**Claim.** \( T \) is a Prüfer domain with quotient field \( K \), \( T = \bigcap_{i=1}^\infty V_i \), and no valuation ring between \( T \) and \( K \) has infinite rank.

**Proof.** Let \( K_n = k(x_1, \ldots, x_n) \), \( R_n = R \cap K_n \), and let \( T_n \) be the integral closure of \( R_n \). Note that for \( s \geq n \), \( V_s \cap K_n = V_{2s} \cap K_n \). Hence

\[
R_n = k + \left( \bigcap_{i=1}^{n-1} M_i \cap K_n \right).
\]

We show that \( T_n = \bigcap_{i=1}^{n-1} V_i \cap K_n \). If \( y \in \bigcap_{i=1}^{n-1} V_i \cap K_n \) then there exists \( a_i \in k \) such that \( y - a_i \in M_i \), for each \( i \) such that \( 1 \leq i < n \). It follows that \( \prod_{i=1}^{n-1} (y - a_i) \in \bigcap_{i=1}^{n-1} M_i \cap K_n \subset R_n \) so \( y \) satisfies an equation of integral dependence over \( R_n \). Thus \( T_n \) is a finite intersection of valuation rings of the field \( K_n \). Hence \( T_n \) is a Prüfer domain with quotient field \( K_n \) and each valuation ring containing \( T_n \) contains some \( V_i \cap K_n \) [1, p. 132–134]. It follows that \( T = \bigcup_{i=1}^\infty T_i \) is also Prüfer [2, p. 260], \( T \) has quotient field \( K \), and \( P = \bigcap_{i=1}^\infty V_i \). Now
suppose \( W \) is a valuation ring between \( T \) and \( K \). Since \( W \) contains \( T_n \), \( W \) contains some \( V_i \cap K_n \). If \( W \) contains \( V_1 \cap K_n \) for all \( n \), then \( W \) contains \( V_1 \) so either \( W = V_1 \) or \( W = K \). If \( V_1 \cap K_n \subseteq W \), then for \( s \geq n \), let \( W_s = W \cap K_s \). We know that \( V_j \cap K_s \subset W_s \) for some \( j < s \). But, for \( j \geq n \), \( V_j \cap K_s \cap K_n = V_1 \cap K_n \), so \( W_s \) is contained in \( V_j \cap K_s \) for some \( j < n \). Since \( V_j \) has rank \( j \), we see that \( W_s \) has rank \( < n \). It follows that \( W = \bigcup_{s=n}^{\infty} W_s \) also has rank less than \( n \).

References