COGENERATOR ENDOMORPHISM RINGS
RONALD L. WAGONER

Abstract. If $R$ is a ring and $P$ is a finitely generated projective right $R$-module, what properties of $R$ does the $R$-endomorphism ring of $P$ inherit? Rosenberg and Zelinsky have shown that if $R$ is quasi-Frobenius, and $P$ also has every simple epimorphic image isomorphic to a submodule, then the $R$-endomorphism ring of $P$ is also quasi-Frobenius. In this paper we show that if $R$ is a cogenerator ring, and $P$ is a finitely generated projective right $R$-module with every simple epimorphic image isomorphic to a submodule of $P$, then the $R$-endomorphism ring of $P$ is also a cogenerator ring.

0. Introduction. If a right $R$-module $P_R$ is a progenerator, and $S = \text{End}_R(P)$, then $R$ and $S$ are categorically equivalent. However, if $P_R$ is just finitely generated projective, surprisingly little is known about $S$.

In this connection, Rosenberg and Zelinsky [5] have shown that if $R$ is quasi-Frobenius and $P_R$ is a finitely generated projective right $R$-module with every simple epimorphic image isomorphic to a simple submodule, then $\text{End}_R(P)$ is also quasi-Frobenius. We call a right $R$-module $M_R$ an $RZ$ module if every simple epimorphic image of $M_R$ is isomorphic to a simple submodule of $M_R$.

In this paper we show

**Theorem.** If $R$ is a cogenerator ring and $P_R$ is a finitely generated projective $RZ$ module, then $\text{End}_R(P)$ is also a cogenerator ring.

1. Cogenerator endomorphism rings. Throughout this paper $R$ will denote an associative ring with identity, and $J$ will denote its Jacobson radical.

We adopt the standard notation that $M_R$ ($R_M$) means $M$ is a right (left) $R$-module, and $N_R < M_R$ means $N_R$ is a submodule of $M_R$. For $I_R < R_R$ and $R_{I'} < R_R$,

$$l_R(I_R) = \{ x \in R \mid xI = 0 \}, \quad r_R(R_{I'}) = \{ x \in R \mid I'x = 0 \}. $$

Presented to the Society, January 23, 1970 under the title Endomorphism rings of projective $RZ$ modules; received by the editors June 29, 1970.

AMS 1969 subject classifications. Primary 1655, 1650, 1640.

**Key words and phrases.** Injective cogenerator, injective envelope, endomorphism ring, cogenerator ring, basic idempotent, finitely generated projective module.

1 This paper is a portion of the author's Ph.D. dissertation written under the supervision of Professor F. W. Anderson and submitted to the graduate faculty of the University of Oregon in June 1969. The author wishes to express his gratitude to Professor Anderson for his encouragement and helpful suggestions.

Copyright © 1971, American Mathematical Society
A ring $R$ is a cogenerator ring if $rR$ and $Rr$ are cogenerators; equivalently, $R$ is a cogenerator ring if $rR$ and $Rr$ are injective and for each $rI < Rr$ and for each $I_k < R$, $rRr$ is $rI$ and $rRr$ is $I_k$. [3].

Onodera [3] shows that if $R$ is a cogenerator ring, then $R$ is semi-perfect. Hence

$$
 rR \simeq \bigoplus_{i=1}^{n} Re_i \quad \text{and} \quad Rr \simeq \bigoplus_{i=1}^{n} e_iR
$$

where $\{e_1, \ldots, e_n\}$ is an orthogonal collection of primitive idempotents. Since a module $rM$ is a cogenerator if, and only if, $rM$ contains a copy of the injective envelope of each simple left $R$-module [4, Lemma 1], $rR$ and $Rr$ contain copies of the injective envelope of each simple left and right $R$-module respectively. Now let $rU$ and $rU'$ be simple and let $E(rU)$ and $E(rU')$ be their injective envelopes. Then $rU \simeq rU'$ if, and only if, $E(rU) \simeq E(rU')$. Hence, a simple counting argument shows that if $R$ is a cogenerator ring and $\{f_1, \ldots, f_k\}$ is a basic set of primitive idempotents for $R$ (for each primitive idempotent $e$ of $R$, $Re$ is isomorphic to exactly one of $Rf_1, \ldots, Rf_k$), then each $Rf_i$ has a simple essential socle and there exists a permutation $\sigma$ of $\{1, \ldots, k\}$ such that

$$
\text{soc}(Rf_i) \simeq Rf_{\sigma(i)}/Jf_{\sigma(i)}.
$$

1.1. **Proposition.** Let $R$ be a cogenerator ring and let $e$ be a primitive idempotent in $R$. Then $\text{soc}(eR) \simeq fR/fJ$ if, and only if, $\text{soc}(eR) \simeq fR/fJ$. Proof. Let $\text{soc}(eR) \simeq fR/fJ$. Then $f$ is also a primitive idempotent. Suppose $Re/Je \simeq \text{soc}(Rg)$ and let $(\_)^*$ denote $\text{Hom}_R(\_ , R)$. Then

$$
Re \to Re/Je \to 0
$$

is exact, and $rR$ is injective, so

$$
0 \to (Re/Je)^* \to (Re)^*
$$

is exact. Hence $\text{soc}(eR) \simeq (Re/Je)^* \text{(duals of simples are simple [2, Theorem 2])}. \text{So (Re/Je)^*} \simeq (\text{soc}(Rg))^* \simeq (\text{soc}(eR))$. Since $0 \to \text{soc}(Rg) \to Rg$ is exact, $(Rg)^* \to (\text{soc}(Rg))^* \to 0$ is also exact. Hence $gR/gJ \simeq (\text{soc}(Rg))^* \simeq (\text{soc}(eR))^* \simeq fR/fJ$ and $gR \simeq fR$ so $Re/Je \simeq \text{soc}(Re)$. By symmetry we get the converse. If $R$ is semiperfect and $P_R$ is finitely generated projective, then $P_R \simeq \bigoplus_{i=1}^{n} e_iR$ with each $e_i$ a primitive idempotent of $R$. In this case
the basic submodule of \( P_R \), denoted by \( B(P) \), is
\[
B(P) = \bigoplus_{i=1}^{t} f_iR
\]
with each \( f_i \in \{ e_1, \ldots, e_m \} \) and for each \( j \in \{ 1, \ldots, m \} \), \( e_jR \) is isomorphic to exactly one of \( f_1R, \ldots, f_tR \). Since \( R \) is semiperfect, the basic submodule is unique up to isomorphism, and is isomorphic to a direct summand of \( R \). We will write \( B(P) = fR \) when \( B(P) \cong fR \) and \( f \) is an idempotent in \( R \). If \( e \) is an idempotent of \( R \) and \( B(eR) = eR \), we will say \( e \) is a basic idempotent.

1.2. Corollary. Let \( R \) be a cogenerator ring and let \( e \) be a basic idempotent in \( R \). Then \( \text{soc}(eR) \cong eR/eJ \) if, and only if, \( \text{soc}(eR) \cong eR/eJ \).

1.3. Proposition. Let \( R \) be a cogenerator ring and let \( P_R \) be finitely generated projective. Then the following are equivalent:
(a) \( P_R \) is an \( RZ \) module.
(b) \( B(P) = eR \) and \( \text{soc}(eR) \cong eR/eJ \).
(c) \( B(P) = eR \) and \( \text{soc}(eR) \cong eR/eJ \).
(d) \( \text{Hom}_R(P, R) \) is an \( RZ \) module.

Proof. (a)\( \iff \) (b): \( P_R \) is an \( RZ \) module if, and only if, \( B(P) = eR \) is an \( RZ \) module. A simple counting argument shows \( eR \) is an \( RZ \) module if, and only if, \( \text{soc}(eR) \cong eR/eJ \).

(b)\( \iff \) (c): 1.2.

(c)\( \iff \) (d): Same as (a)\( \iff \) (b), since \( B(P) = eR \) if, and only if, \( \text{Hom}_R(P, R) = eR \).

1.4. Theorem. Let \( R \) be a cogenerator ring and let \( P_R \) be a finitely generated projective \( RZ \) module. Then \( \text{End}_R(P) \) is also a cogenerator ring.

Proof. Let
\[
P \cong \bigoplus_{i=1}^{n} e_iR \quad \text{and} \quad B(P) = eR = e_1R \oplus \cdots \oplus e_tR
\]
with each \( e_i \) a primitive idempotent.

By [1, Theorem 1.5] \( eRe \) and \( \text{End}_R(P) \) are categorically equivalent, hence we need only see that \( eRe \) is a cogenerator ring.

Now, \( eRe \cong \bigoplus_{i=1}^{t} eRe_i \) and each \( eRe_i \) is indecomposable since
\[
eRe_i \cong eR \otimes Re_i
\]
and
Let $0 \neq eM \leq eRe_i$, then $0 \neq ReM \leq Re_i = eRe_i$. Hence $soc(Re_i) \leq ReM$ and so $e \cdot soc(Re_i) \leq eReM = eM$. Since $soc(Re) \approx Re/Je$, $e \cdot soc(Re_i) \neq 0$. Hence, for each $i = 1, \ldots, k$, $e \cdot soc(Re_i)$ is a simple essential submodule of $eRe_i$. Let $E[e \cdot soc(Re_i)]$ be the injective envelope of $e \cdot soc(Re_i)$, then

$$eRe \leq \bigoplus_{i=1}^{k} E[e \cdot soc(Re_i)] \leq \prod_{A} eR.$$  

(If $R$ is a cogenerator then $e_{Re}eRe$ is also a cogenerator since $0 \rightarrow Re \otimes eM \rightarrow \prod R$ exact, gives

$$0 \rightarrow eR \otimes Re \otimes eM \rightarrow eR \otimes \prod R$$

exact, and $eR \otimes Re \otimes eM \approx eM$ and $eR \otimes \prod R \approx \prod eR$.)

Let $e = (e_{\alpha})_{\alpha \in A}$ and let $L_{R}$ be the submodule of $e_{R}$ generated by $\{e_{\alpha} | \alpha \in A\}$. Then let $f \in \text{Hom}_{R}(e_{R}/L, R)$. Now, $e_{R}/L = (e+L)e_{R}$ and $L = e_{L} + L = (e+L)e_{L}$ so $0 = f(0) = f(e+L)e_{L}$ hence $e_{R} \cdot f(e+L)e_{L} = 0$. But then $e_{R} \cdot f(e+L)e = 0$ in $\prod e_{R}$, so $e_{R} \cdot f(e+L)e = 0$. Since $soc(Re) \approx Re/Je$, $R \cdot f(e+L)e = 0$, so $f(e+L)e = 0 = f(e+L)$ and so $f = 0$. Now, $\text{Hom}_{R}(e_{R}/L, R) = 0$ and $R$ is a cogenerator, so $e_{R} = L$. Hence there exist elements $x_{1}, \ldots, x_{m}$ in $R$ such that

$$\sum_{i=1}^{m} er_{i}x_{i} = e.$$  

Let $\pi_{i}$ be the projection of $\prod_{A} e_{R}$ onto the $i$th coordinate then

$$\sum_{i=1}^{m} \pi_{i}x_{i}: \prod \limits_{A} e_{R} \rightarrow eRe$$

via

$$(e_{\gamma})_{\alpha \in A} \rightarrow \sum_{i=1}^{m} e_{\gamma}x_{i}e$$

splits the embedding of $eRe$ in $\prod_{A} e_{R}$. Hence $eRe$ is a direct summand of $\bigoplus_{i=1}^{k} E[e \cdot soc(Re_i)]$ and so is injective and contains a copy of each simple left $eRe$-module. Hence, $e_{Re}eRe$ is an injective cogenerator.

Now, using Proposition 1.3, we can repeat the above arguments.
on the opposite side with $Re$, and get $eRe_{eRe}$ is an injective cogenerator.

**Bibliography**


Fresno State College, Fresno, California 93726