A NOTE ON TWO-SIDED IDEALS IN $C^*$-ALGEBRAS\(^1\)

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Abstract. An elementary proof is given of the fact that $(I+J)^+ = I^+ + J^+$ for $I$ and $J$ closed two-sided ideals in a $C^*$-algebra.

In this note we give a short and elementary proof of the following theorem, which was proposed as a problem by J. Dixmier [1, Problem 1.9.12] and first proved by E. Störmer [4].

Theorem. Let $A$ be a $C^*$-algebra. If $I$ and $J$ are uniformly closed two-sided ideals in $A$ then $(I+J)^+ = I^+ + J^+$.

Here $I^+$ denotes the set of positive elements in $I$. To prove the theorem we may assume that $A$ has an identity, denoted by $e$. In [4] Störmer proved the theorem by using some results of E. G. Effros [2] and some quite delicate calculations using the functional calculus to prove the following lemma, from which the theorem follows by a straightforward induction argument.

Lemma. With the assumptions as in the theorem, let $a$ belong to $(I+J)^+$, and let $e > 0$ be given. Then there exists $b$ in $I^+$ and $c$ in $J^+$ such that $0 \leq a - b - c \leq ee$.

A SHORT PROOF OF THE LEMMA. Let $a$ belong to $(I+J)^+$. Then $a = f + g$ for some $f$ in $I$ and $g$ in $J$. Since $a = a^*$, we may assume that $f = f^*$ and $g = g^*$. Let $h = |f| + |g| + ee$. Then $h$ is invertible and $0 \leq a - h$. Let $d = a^{1/2}h^{-1/2}$. Then $d$ is in $A$, and $0 \leq d^*d = h^{-1/2}ah^{-1/2} \leq h^{-1/2}hh^{-1/2} = e$. Thus $dd^* \leq e$, and $0 \leq a - d|f|d^* - d|g|d^* = edd^* \leq ee$. Since $d|f|d^*$ is in $I^+$ and $d|g|d^*$ is in $J^+$ the lemma is proved.

We note that G. K. Pedersen has given another proof of this theorem [3].

References


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