A CHARACTERIZATION OF A SPACE WITH COUNTABLE INFINITY

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ABSTRACT. It is well known that, for a countable discrete space N,

$$|\beta N - N| = 2^{2^{\aleph_0}}.$$

So, any completely regular T_1 space X with $|\beta X - X| \leq \aleph_0$ does not contain any infinite discrete subspace. In this paper, we characterize those completely regular T_1 spaces with countable infinity as follows: Such a space X is characterized by the two properties.
    (a) X is pseudocompact.
    (b) There exist a compact metric space Y and a continuous map f from X onto Y so that the subset $Y_0 = \{y : f^{-1}(y)$ is not compact$\}$ of Y is countable and $\mathrm{cl}_{\beta X} f^{-1}(y) - f^{-1}(y)$ is one point whenever $y \in Y_0$. (In particular, for any y in Y_0, $\beta(f^{-1}(y)) - f^{-1}(y)$ is one point if X is normal.)

1. **Introduction.** Let βX denote the Stone-Čech compactification of a completely regular T_1 space X, let $|A|$ denote the cardinal number of a set A, and finally, let us call the set $\beta X - X$ the *infinity*. It is well known that $|\beta N - N| = 2^{2^{\aleph_0}}$ for the countable discrete space N (cf. [1]). Thus, in general, $|\beta X - X|$ is large. But, on the other hand, there exist spaces with the infinity consisting of only one point (for example, the space of all countable ordinal numbers with usual topology), and if we form the product of such a space with a compact T_1 space consisting of countably many points, we obtain a space with the infinity consisting of countably many points (for brevity, we say *countable infinity*).

A completely regular T_1 space with one-point infinity has been characterized in [1, 6J] and a locally compact, regular T_1 space with finite or countable compactification (that is, a space with finite or countable infinity for some compactification) has been discussed and characterized in [3] and [4].

The purpose of this paper is to characterize a completely regular T_1 space with countable infinity through the use of a suitable compact metric space and a continuous map. Also, a completely regular T_1 space with finite infinity is characterized as a special case.

THEOREM 1.[1] *For each completely regular T_1 space X, the following conditions are equivalent*:

(1) $|\beta X - X| \leq \aleph_0$,

(2) *X is pseudocompact and there exists a continuous map f from X onto a compact metric space Y so that the subset*

$$Y_0 = \{y \in Y : f^{-1}(y) \text{ is not compact}\}$$

is countable and $|\mathrm{cl}_{\beta X} f^{-1}(y) - f^{-1}(y)| = 1$ *whenever* $y \in Y_0$. *(In particular,* $|\beta(f^{-1}(y)) - f^{-1}(y)| = 1$ *for* $y \in Y_0$ *if X is normal.)*

(3) *For any continuous map f from X onto an arbitrary compact metric space Z the subset*

$$Z_0 = \{z \in Z : f^{-1}(z) \text{ is not compact}\}$$

of Z is countable and $|\mathrm{cl}_{\beta X} f^{-1}(y) - f^{-1}(y)| \leq \aleph_0$ *whenever* $z \in Z_0$. *(In particular,* $|\beta(f^{-1}(z)) - f^{-1}(z)| \leq \aleph_0$ *for* $z \in Z_0$ *if X is normal.)*

As a special case, for a space with finite infinity we have Theorem 2.

THEOREM 2. *For each completely regular T_1 space X the following conditions are equivalent*:

(1) $|\beta X - X| = n$ *(a positive integer).*

(2) *X is pseudocompact and there exists a continuous map f from X onto a compact metric space Y so that the subset* $Y_0 = \{y \in Y : f^{-1}(y)$ *is not compact*$\}$ *consists of just n points and* $|\mathrm{cl}_{\beta X} f^{-1}(y) - f^{-1}(y)| = 1$ *whenever* $y \in Y_0$. *(In particular,* $|\beta(f^{-1}(y)) - f^{-1}(y)| = 1$ *for* $y \in Y_0$ *if X is normal.)*

The proofs that (1) and (2) are equivalent in Theorems 1 and 2 are similar and therefore we only prove Theorem 1.

2. **Proof of Theorem 1.** (1) *implies* (2). First, the condition (1) implies the pseudocompactness of X. Because, if X were not pseudo-compact, it would contain a closed subset F of X which could be saturated with a countable (infinite) discrete subspace N of the real line R by a continuous function $f : X \to R$; that is, $f(F) = N$ and $f^{-1}(N) = F$. Then we have $\beta(f)^{-1}(\mathrm{cl}_{\beta R} N - N) \subset \beta X - X$, where $\beta(f)$ denotes the extension of f over βX. Since $\beta N = \mathrm{cl}_{\beta R} N$ holds (cf. [1]) and N has an uncountable infinity, $\beta X - X$ must be uncountable, which contradicts to the condition (1).

Next, let us put $\beta X - X = \{p_1, p_2, \cdots\}$, where $p_i \neq p_j$ for any $i \neq j$. Since βX is a completely regular T_1 space, for any distinct m, n

there exists a continuous map f_{mn} from βX into I (the unit closed interval) with values $f_{mn}(p_m) = 0$ and $f_{mn}(p_n) = 1$. Define

$$e: \beta X \to P = \prod_{m,n} I \quad \text{(a metric space)}$$

by $(e(x))_{m,n} = f_{mn}(x)$, and put $f = e | X$ and $Y = f(X)$. Then Y is a compact metric space. Moreover, $Y = e(\beta X)$. This follows from the fact that the continuous image of a pseudocompact space is pseudocompact, any pseudocompact subspace of a metric space is compact (and therefore closed) and Y is also dense in $e(\beta X)$. Furthermore, since X is pseudocompact and each point of Y is G_δ, f is a Z-map and hence a WZ-map; i.e., $\mathrm{cl}_{\beta X}(f^{-1}(y)) = e^{-1}(y)$ for each $y \in Y$ (cf. [2]). Let us put $q_i = e(p_i)$ for $i = 1, 2, \cdots$. Then for any $y \in Y$ which is distinct from any q_i we have $e^{-1}(y) \subset X$ and so $f^{-1}(y) = e^{-1}(y)$ is compact. For q_i, from the definition of e, we can easily see that q_i's are mutually distinct and so we have $e^{-1}(q_i) = f^{-1}(q_i) \cup \{p_i\}$. This completes the proof in the case that X is a completely regular T_1 space. If X is normal, we can assert $\beta(f^{-1}(q_i)) = \mathrm{cl}_{\beta X} f^{-1}(q_i)$ for $i = 1, 2, \cdots$. Thus the proof is completed.

(2) *implies* (1). Let X, Y, f and Y_0 be given as in (2). By the same reasoning as before, we have $\mathrm{cl}_{\beta X}(f^{-1}(y)) = \beta(f)^{-1}(y)$ for each $y \in Y$. For each $y_i \in Y_0$ we denote $\mathrm{cl}_{\beta X} f^{-1}(y_i) = f^{-1}(y_i) \cup \{p_i\}$ for some $p_i \in \beta X - X$. Of course, βX is the complete inverse image of Y by $\beta(f)$ and therefore we have $\beta X = \cup \{\beta(f)^{-1}(y) : y \in Y\}$ $= \cup \{\mathrm{cl}_{\beta X} f^{-1}(y) : y \in Y\} = \cup \{f^{-1}(y) : y \in Y\} \cup \{p_1, p_2, \cdots\}$. This completes the proof.

(1) *is equivalent to* (3). The proof is almost clear. For the direction (1)$\Rightarrow$(3), if we put $\beta X = X \cup \{p_1, p_2, \cdots\}$, then $Z_0 = \{\beta(f)(p_i) : i = 1, 2, \cdots\}$ satisfies the required conditions. (Notice that again f is a WZ-map.) For the direction (3)$\Rightarrow$(1), it suffices to let Z be a one-point space.

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