SEMIGROUPS ON ACYCLIC PLANE CONTINUUM

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Abstract. It is shown that an acyclic irreducible plane continuum which admits the structure of a topological semigroup is an arc if it has an identity, and is either an arc, is trivial, or is decomposible into an arc if it satisfies $M^2 = M$. This extends some results of Friedberg and Mahavier concerning semigroups on chainable continua.

Let $M$ be a topological semigroup with minimal ideal $K$ whose underlying space is a nondegenerate compact metric continuum. If $M$ has an identity, $M$ is called a clan.

Under the assumption that $M$ is chainable, Friedberg and Mahavier [3] showed that if $M$ is a clan it is an arc, and if $M^2 = M$ then either $M$ is trivial, $M$ is an arc, or $M|K$ is an arc and $M$ is irreducible from a one-sided identity to some point. In this note we extend these results (using essentially the same arguments) by replacing the condition that $M$ be chainable by the condition that $M$ be an acyclic (i.e., contains no simple closed curve) plane continuum which is irreducible between two points. (Every nondegenerate chainable continuum is homeomorphic to such a continuum.)

Theorem 1. If $M$ is an acyclic clan in the plane, then $M$ is arcwise connected.

Proof. Let $G$ be a closed subgroup of $M$ with identity $e$ and let $C(e)$ be the component of $G$ containing $e$. $C(e)$ is a subcontinuum of $M$ and is a group. Suppose $C(e)$ is nondegenerate. Then it is homogeneous and by [4] contains an arc; so by [1] it is a simple closed curve, contradicting the assumption that $M$ be acyclic. Thus $C(e)$ is degenerate and $G$ is totally disconnected. Then $M$ is arcwise connected by [6].

Corollary. If $M$ is an acyclic plane continuum which is irreducible between two of its points it is an arc.

Remark. The referee has observed that except for the existence of the one-sided identity, the conclusion of the next theorem follows from Hunter's argument in [5, Theorem 8], without the assumption.
that \( M \) be acyclic. Also, a simplification suggested by the referee has been employed in the next argument.

**Theorem 2.** If \( M \) is an acyclic plane continuum which is irreducible between two of its points and \( M^2 = M \), then either

1. \( M = K \) and the multiplication on \( K \) is trivial,
2. \( M \) is an arc, or
3. \( M \) has a one-sided identity \( e \), \( M \mid K \) is an arc, and \( M \) is irreducible from \( e \) to some point.

**Proof.** Let \( E \) denote the set of idempotent elements of \( M \), and for \( e \) in \( E \), let \( H_e \) be the maximal subgroup containing \( e \). Since \( M \) is acyclic, \( K \) is not the cartesian product of two nondegenerate continua [5, Lemma 2, p. 238]; so \( K \) is a group or multiplication in \( K \) is trivial [7, Corollary 1]. As in the proof of Theorem 1, if \( K \) is a group it is degenerate. In either case multiplication in \( K \) is trivial and \( K \) is a subset of \( E \).

Now assume that \( M \not\subseteq K \) and \( M \) is not an arc. Suppose \( M \) has no one-sided identity. Since \( M \) is irreducible between two points \( a \) and \( b \), there exist points \( e \) and \( f \) in \( E \setminus K \) such that \( a \in H_e \), \( b \in H_f \), \( H_e \) and \( H_f \) are connected, and \( M = (eMe) \cup (fMf) \) [7, Theorem 5]. But \( H_e \) and \( H_f \) are degenerate so \( M \) is irreducible from \( e \) to \( f \). Since \( eMe \) and \( fMf \) are acyclic plane clans, they are arcwise connected by Theorem 1. Then \( M \) is an arc from \( e \) to \( f \), a contradiction. Thus \( M \) has a right (or left) identity \( e \).

Then \( Me = M \) and \( eM = eMe \) is either degenerate or arcwise connected. If \( eM \) is degenerate, \( e \in K \) and \( Me = M = K \), a contradiction. Hence \( eM = eMe \) is a nondegenerate arcwise connected clan with \( e \) as its identity. Let \( T \) be an arc in \( eM \) from \( e \) to its minimal ideal \( K' \) such that \( T \cap K' \) is degenerate. Clearly \( K' \subseteq K \). Since each of \( aT \) and \( bT \) is a continuous image of \( T \), each is either degenerate or arcwise connected, and there is an \( \alpha \) and a \( \beta \) such that each of \( \alpha \) and \( \beta \) is an arc or degenerate, \( \alpha \subseteq aT \), \( \beta \subseteq bT \), \( \alpha \) contains \( a \), \( \beta \) contains \( b \) and each of \( \alpha \) and \( \beta \) intersects \( K \) at only one point. Since \( M \) is irreducible from \( a \) to \( b \), \( M = a \cup K \cup \beta \). If both \( a \) and \( b \) belong to \( K \), \( K = M \), so let \( e \in \beta \setminus K \). If \( e \not\in b \), \( e \) possesses a euclidean (1-dimensional) neighborhood and since \( e \) is a right identity, \( e \in K \), a contradiction [2, Lemma 4]. Hence \( e = b \) and (3) holds.

**Remark.** An application of Theorem 1 to some nonchainable continua would be as follows: no continuum in the plane consisting of an infinite half-ray “spiraling down” upon a nondegenerate acyclic continuum admits the structure of a topological semigroup with identity.
References


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