ON SOME PRODUCTS INVOLVING PRIMES

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ABSTRACT. Asymptotic formulae are given for the products $P_l(x)$ $(l = 1, 3)$ defined below.

We put, for $x \geq 2$ and $l = 1$ and 3,

$$P_l(x) = \prod_{p \leq x; \, p \equiv l(4)} \left(1 - \frac{1}{p}\right),$$

where the product is taken over the specified primes p. Our aim in the present note is to show that

(1)
$$P_1(x) = (\pi A_1 e^{-C})^{1/2}(\log x)^{-1/2} + O((\log x)^{-3/2}),$$
$$P_3(x) = \left(\frac{\pi A_3 e^{-C}}{2}\right)^{1/2}(\log x)^{-1/2} + O((\log x)^{-3/2}),$$

where C denotes the Euler constant and

$$A_l = \prod_{p \equiv l(4)} \left(1 - \frac{1}{p^2}\right)$$

(so that $A_1 A_3 = 8/\pi^2$).

Now, let us define $\chi(n) = 0$ for even n and $= (-1)^{(n-1)/2}$ for odd n. Then, $\chi(n)$ is a residue character (mod 4), and the corresponding L-series $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ represents a continuous function of s for $s > 0$. In particular, we have $L(1, \chi) = \pi/4$ and

$$L(1, \chi) = \prod_{p \leq x} \left(1 - \frac{\chi(p)}{p}\right)^{-1} + O\left(\frac{1}{\log x}\right)$$

(cf. [1, §109]), whence

(2)
$$\frac{P_3(x)}{P_1(x)} = \frac{\pi}{4} A_3 + O\left(\frac{1}{\log x}\right).$$

On the other hand, we have by a well-known theorem due to F. Mertens (cf. [1, §36])

(3)
$$P_1(x) P_3(x) = \frac{2e^{-C}}{\log x} + O\left(\frac{1}{\log^2 x}\right).$$

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The result (1) follows at once from (2) and (3).

We note that our asymptotic formula for $P_1(x)$ will give a solution to a problem recently posed by D. Suryanarayana in [2].

REFERENCES

1. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen.* Vols. 1, 2, 2nd ed., Chelsea Publishing Co., New York, 1953. MR 16, 904.
2. D. Suryanarayana, *Problem* 17 (1), Bull. Amer. Math. Soc. **76** (1970), 976.

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