ON THE CONJUGACY OF INJECTORS

GRAHAM A. CHAMBERS

Abstract. In their paper, *Injektoren endlicher auflösbarer Gruppen*, Fischer, Gaschütz and Hartley ask the following question. If \( \mathcal{F} \) is a normal subgroup closed class of groups and if \( G \) is a finite solvable group which possesses \( \mathcal{F} \)-injectors, is it true that any two \( \mathcal{F} \)-injectors of \( G \) are conjugate in \( G \)? A partial answer is given. It is proven that if \( G \) has \( p \)-length 1 for each prime \( p \), then the answer to this question is yes.

1. Introduction. Fitting classes and injectors were introduced by Fischer, Gaschütz and Hartley [2]. A Fitting class \( \mathcal{F} \) is an isomorphism closed class of groups satisfying \( f_1: G \in \mathcal{F}, N \triangleleft G \) implies \( N \in \mathcal{F} \), \( f_2: N_1, N_2 \triangleleft G, N_1, N_2 \in \mathcal{F} \) implies \( N_1 N_2 \in \mathcal{F} \). If \( G \) is a group, \( V \in G \) is an \( \mathcal{F} \)-injector of \( G \) provided \( V \triangleleft G \) implies \( V \cap N \) is \( \mathcal{F} \)-maximal in \( N \). Satz 1 [2] states that if \( \mathcal{F} \) is a Fitting class and \( G \) is a finite solvable group, then \( G \) possesses \( \mathcal{F} \)-injectors and any two are conjugate. At the close of [2] the authors ask if the conjugacy of injectors can be proven using only the first of the defining properties of a Fitting class. That is, if \( \mathcal{F} \) is an isomorphism closed class of groups satisfying \( f_1 \) and if \( G \) is a finite solvable group which possesses \( \mathcal{F} \)-injectors, is it true that any two \( \mathcal{F} \)-injectors of \( G \) are conjugate? A partial answer is given. We prove that if \( G \) has \( p \)-length 1 for each prime \( p \), then the answer to this question is yes.

2. \( p \)-normally embedded subgroups. In proving our result we will use the concept of a \( p \)-normally embedded subgroup. \( V \triangleleft G \) is said to be \( p \)-normally embedded in \( G \) if a Sylow \( p \)-subgroup \( V_p \) of \( V \) is also Sylow in some normal subgroup of \( G \). This concept was introduced by Hartley [3] and has also been studied in [1]. We are going to need the following theorem which is essentially a restatement of Theorem 2.6 of [1].

**Theorem 1.** Let \( G \) be a finite solvable group and suppose \( V \triangleleft G \) is \( p \)-normally embedded in \( G \) for each prime \( p \). Suppose \( W \triangleleft G \) and that for each prime \( p \) the Sylow \( p \)-subgroups of \( W \) are conjugate to those of \( V \). Then \( V \) and \( W \) are conjugate.

We are also going to need the following theorem which will be used...
to show that if $G$ has $p$-length 1 for each prime $p$, then the $G$-injectors of $G$ are $p$-normally embedded in $G$.

**Theorem 2.** Let $p$ be a prime and let $G$ be a $p$-solvable finite group. Then $G$ has $p$-length 1 if and only if each $p$-subgroup of $G$ is Sylow in some subnormal subgroup of $G$.

**Proof.** Suppose $G$ has $p$-length 1 and that $P$ is a $p$-subgroup of $G$. Let $K = O_p(G)$ and consider $G/K$. $PK/K$ is a $p$-subgroup of $G/K$ and, if $K \neq 1$, $PK/K$ is Sylow in some $L/K < G/K$ by induction. But then $P$ is Sylow in $L < G$ as required. Thus we may assume $K = 1$. Then $G$ has a normal Sylow $p$-subgroup $P^*$ and $P < P^* < G$ so that $P < G$.

To prove the converse we suppose each $p$-subgroup of $G$ is Sylow in some subnormal subgroup of $G$. If $N < G$ and $P/N$ is a $p$-subgroup of $G/N$, then there is a $p$-subgroup $P^*$ of $G$ such that $P = P^*N$. By assumption $P^*$ is Sylow in some $L < G$ so that $P/N = P^*N/N$ is Sylow in $LN/N < G/N$. Thus by induction $G/N$ has $p$-length 1 for any $1 \neq N < G$. If $O_p(G) \neq 1$, then we are done. Otherwise we can assume $G$ has a unique minimal normal subgroup $K$ which is a $p$-group. If $\Phi(G) \neq 1$, then $G/\Phi(G)$ has $p$-length 1 and hence so does $G$. Thus we may assume $\Phi(G) = 1$ so that $K$ is complemented. Assume $MK = G$ and $M \cap K = 1$. If $M$ is $p'$, then $K$ is Sylow $p$ in $G$ and we are done. Suppose then that $1 \neq M_p$ is Sylow $p$ in $M$. By assumption $M_p$ is also Sylow in some $L < G$. Since $K$ is a $p$-group and $M_p \cap K = 1$, $L$ is a proper subgroup of $G$. But then there exists a proper normal subgroup $L^*$ of $G$ such that $M_p \leq L \leq L^*$. Since $K$ is the unique minimal normal subgroup of $G$, $K \leq L^*$. Then $M_pK \leq L^*$ so that $L^*$ has $p'$ index. Now each $p$-subgroup of $L^*$ is Sylow in some $R < G$ and so is Sylow in $L^* \cap R < L^*$. Thus $L^*$ has $p$-length 1 by induction. Since $L^*$ has $p'$ index this implies $G$ has $p$-length 1. Q.E.D.

3. The main theorem.

**Theorem 3.** Suppose $G$ has $p$-length 1 for each prime $p$ and suppose $V$ and $W$ are $G$-injectors of $G$ where $G$ is an isomorphism closed class of groups satisfying $f$. Then

1. $V$ is $p$-normally embedded in $G$ for each prime $p$.
2. $V$ and $W$ are conjugate.

**Proof.** The proof is by induction on $|G|$. We assume both statements have been shown to hold whenever $|G| < n$. Now assume $|G| = n$. Our first step is to show that $|V| = |W|$. Let $M$ be a maximal normal subgroup of $G$. $V \cap M$ and $W \cap M$ are each $G$-injectors of
Let $V_p$ and $W_p$ denote Sylow $p$-subgroups of $V$ and $W$ respectively. Our second step is to show that $V_p$ and $W_p$ are conjugate. If both $V_p$ and $W_p$ are Sylow in $G$, this is clear. Suppose then that $V_p$ is not Sylow in $G$. From Theorem 2 we know $V_p$ is Sylow in some proper subnormal subgroup $L$ of $G$. $V \cap L$ and $W \cap L$ are each $\mathfrak{F}$-injectors of $L$ and so they are conjugate by induction. Choose $g$ such that $V \cap L = (W \cap L)^g \leq W^g$. Then $V_p$ is Sylow in $V \cap L \leq W^g$ so that $V_p$ is contained in some conjugate of $W_p$. Since $V$ and $W$ have the same order so do $V_p$ and $W_p$ and so we conclude that $V_p$ and $W_p$ are conjugate.

The next step is to show that $V$ is $\mathfrak{F}$-normally embedded in $G$. By Theorem 2, $V_p$ is Sylow in some $L \triangleleft G$. If $L = G$, then $V_p$ is Sylow in $G$ so that $V$ is $\mathfrak{F}$-normally embedded in $G$. If $L$ is proper then there is a proper normal subgroup $H$ of $G$ such that $V_p \leq L \leq H$. $V \cap H$ is an $\mathfrak{F}$-injector of $H$ and $V_p$ is Sylow in $V \cap H$. Since $H \triangleleft G$, $V \cap H$ is $\mathfrak{F}$-normally embedded in $H$ by induction. That is, $V_p$ is Sylow in some normal subgroup $K$ of $H$. But then $V_p$ is Sylow in $(V_p)^H \leq K$. Suppose now that $\alpha \in \text{Aut}(H)$. Then $(V \cap H)^\alpha$ is again an $\mathfrak{F}$-injector of $H$ and since $|H| < |G|$, $(V \cap H)^\alpha$ is conjugate to $V \cap H$ in $H$ by induction. In particular $(V_p)^\alpha$ is conjugate to $V_p$ in $H$. This shows that $(V_p)^H$ is in fact characteristic in $H \triangleleft G$. But then $V_p$ is Sylow in $(V_p)^H \triangleleft G$ so that $V$ is $\mathfrak{F}$-normally embedded in $G$ as required.

As a final step we invoke Theorem 1 to complete the proof that $V$ and $W$ are conjugate. Q.E.D.

References


University of Alberta, Edmonton, Alberta, Canada