EVERY ISOMETRY IS REFLEXIVE

JAMES A. DEDDENS[1]

ABSTRACT. A bounded linear operator A on a Hilbert space $\mathfrak{IC}$ is called reflexive if any bounded linear operator which leaves invariant the invariant subspaces of A is a limit of polynomials in A in the weak operator topology. In this note we prove that every isometry V on a Hilbert space $\mathfrak{IC}$ is reflexive.

Let A be a bounded linear operator on a Hilbert space $\mathfrak{IC}$. We denote by $\mathrm{Lat}(A)$ the set of all closed subspaces $\mathfrak{M}$ of $\mathfrak{IC}$ such that $A\mathfrak{M} \subseteq \mathfrak{M}$, and by $\mathfrak{a}_A$ the smallest weakly closed algebra containing A and I (i.e. the closure in the weak operator topology of $p(A)$ for all polynomials p). A is called *reflexive* if, for B a bounded linear operator on $\mathfrak{IC}$, $\mathrm{Lat}(A) \subseteq \mathrm{Lat}(B)$ implies $B \in \mathfrak{a}_A$.

D. Sarason in [4] proved that every normal (and hence every unitary) operator is reflexive, and that the unilateral shift of multiplicity one is also reflexive. Our task is to piece together these two results to show that every isometry is reflexive. We might mention that it is unknown whether the direct sum of two arbitrary reflexive operators is reflexive.

Let V be an isometry on $\mathfrak{IC}$ (i.e. $\|Vx\| = \|x\|$ for all $x \in \mathfrak{IC}$). Then there exist unique reducing subspaces $\mathfrak{M}_\infty \equiv \bigcap_{n=0}^{\infty} V^n\mathfrak{IC}$ and $\mathfrak{M}_+ \equiv \sum_{n=0}^{\infty} \oplus V^n(\mathfrak{IC} \ominus V\mathfrak{IC})$ such that $\mathfrak{IC} = \mathfrak{M}_\infty \oplus \mathfrak{M}_+$ with $U = V|_{\mathfrak{M}_\infty}$ a unitary operator and $U_+ = V|_{\mathfrak{M}_+}$ a unilateral shift [3, Problem 118]. If we let $E(\cdot)$ be the spectral measure of U and $\mathfrak{M}_s \equiv \{x \in \mathfrak{M}_\infty : \|E(\cdot)x\| \perp \mathrm{Leb}\}$ and $\mathfrak{M}_a \equiv \{x \in \mathfrak{M}_\infty : \|E(\cdot)x\| \ll \mathrm{Leb}\}$ then $\mathfrak{M}_\infty \equiv \mathfrak{M}_s \oplus \mathfrak{M}_a$ with $U_s = U|_{\mathfrak{M}_s}$ and $U_a = U|_{\mathfrak{M}_a}$ being called the singular and absolutely continuous parts of U respectively [2].

THEOREM. *Every isometry* $V = U_s \oplus U_a \oplus U_+$ *on a Hilbert space* $\mathfrak{IC}$ *is reflexive.*

Before we prove this result we need some lemmas, which are of interest in their own right.

LEMMA 1. $\mathrm{Lat}(U_s \oplus U_a) = \mathrm{Lat}(U_s) \oplus \mathrm{Lat}(U_a)$.

PROOF. We need only show that $\mathfrak{M} \in \mathrm{Lat}(U_s \oplus U_a)$ implies $\mathfrak{M} = P_{\mathfrak{M}_s}\mathfrak{M} \oplus P_{\mathfrak{M}_a}\mathfrak{M}$. First suppose $\mathfrak{M}$ reduces $U_s \oplus U_a$. Then $P_{\mathfrak{M}}$ commutes with $U_s \oplus U_a$ or matricially,

$$P_{\mathfrak{M}} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad \text{commutes with} \quad \begin{pmatrix} U_s & 0 \\ 0 & U_a \end{pmatrix}.$$

So $BU_a = U_s B$. By Theorem 3 in [1] we note that $B = 0$. Hence $P_{\mathfrak{M}} = A \oplus C$ or $\mathfrak{M} = P_{\mathfrak{M}_s}\mathfrak{M} \oplus P_{\mathfrak{M}_a}\mathfrak{M}$. Now suppose $\mathfrak{M} \in \mathrm{Lat}(U_s \oplus U_a)$ with $U = U_s \oplus U_a$. If $\mathcal{L} = \mathfrak{M} \ominus U\mathfrak{M}$ then

$$\mathfrak{M} = \sum_{n=0}^{\infty} \oplus \ U^n\mathcal{L} \ \oplus \ \bigcap_{n=0}^{\infty} U^n\mathfrak{M} \equiv \mathfrak{N} \ \oplus \ \mathfrak{R},$$

where $\mathfrak{R}$ reduces U. Hence $\mathfrak{R} = P_{\mathfrak{M}_s}\mathfrak{R} \oplus P_{\mathfrak{M}_a}\mathfrak{R}$. Since

$$\mathfrak{N} = \sum_{n=0}^{\infty} \oplus \ U^n\mathcal{L} \subseteq \sum_{n=-\infty}^{\infty} \oplus \ U^n\mathcal{L},$$

and since the bilateral shift is an absolutely continuous unitary, we must have that $\mathfrak{N} \subseteq \mathfrak{M}_a$, so $\mathfrak{N} = P_{\mathfrak{M}_s}\mathfrak{N} \oplus P_{\mathfrak{M}_a}\mathfrak{N}$. Thus $\mathfrak{M} = \mathfrak{N} \oplus \mathfrak{R} = P_{\mathfrak{M}_s}\mathfrak{M} \oplus P_{\mathfrak{M}_a}\mathfrak{M}$.

LEMMA 2. $\mathfrak{A}_{U_s \oplus U_a} = \mathfrak{A}_{U_s} \oplus \mathfrak{A}_{U_a}$.

PROOF. We need only show that $B_1 \in \mathfrak{A}_{U_s}$ and $B_2 \in \mathfrak{A}_{U_a}$ imply $B_1 \oplus B_2 \in \mathfrak{A}_{U_s \oplus U_a}$. Since $B_1 \in \mathfrak{A}_{U_s}$ and $B_2 \in \mathfrak{A}_{U_a}$, $\mathrm{Lat}(U_s) \subseteq \mathrm{Lat}(B_1)$ and $\mathrm{Lat}(U_a) \subseteq \mathrm{Lat}(B_2)$. Hence $\mathrm{Lat}(U_s) \oplus \mathrm{Lat}(U_a) \subseteq \mathrm{Lat}(B_1 \oplus B_2)$. Now Lemma 1 implies that $\mathrm{Lat}(U_s \oplus U_a) \subseteq \mathrm{Lat}(B_1 \oplus B_2)$. Since $U_s \oplus U_a$ is unitary, it is reflexive by Sarason's result [4], and hence $B_1 \oplus B_2 \in \mathfrak{A}_{U_s \oplus U_a}$.

COROLLARY. $\mathfrak{A}_{U_s \oplus U_a \oplus U_+} = \mathfrak{A}_{U_s} \oplus \mathfrak{A}_{U_a \oplus U_+}$.

PROOF. We need only show that $B_1 \in \mathfrak{A}_{U_s}$ and $B_2 \in \mathfrak{A}_{U_a \oplus U_+}$ imply $B_1 \oplus B_2 \in \mathfrak{A}_{U_s \oplus (U_a \oplus U_+)}$. Let W be the bilateral shift on $\mathfrak{R}_+$ which uniquely extends the unilateral shift U_+, i.e. $\mathfrak{R}_+ = \bigvee_{n=-\infty}^{\infty} W^n\mathfrak{M}_+$ and $U_+ = W|_{\mathfrak{M}_+}$. Then there is a unique $\tilde{B}_2 \in \mathfrak{A}_{U_a \oplus W}$ such that $B_2 = \tilde{B}_2|_{\mathfrak{M}_a \oplus \mathfrak{M}_+}$, in fact if $B_2 = \psi(U_a \oplus U_+)$ for $\psi \in H^\infty$ then $\tilde{B}_2 = \psi(U_a \oplus W)$. Since $U_a \oplus W$ is an absolutely continuous unitary, Lemma 2 implies that $B_1 \oplus \tilde{B}_2 \in \mathfrak{A}_{U_s \oplus (U_a \oplus W)}$. Hence $B_1 \oplus B_2 \in \mathfrak{A}_{U_s \oplus (U_a \oplus U_+)}$.

LEMMA 3. $U_a \oplus U_+$ is reflexive.

PROOF. The proof can be simplified if we make the following observation: If A is reflexive and $\{\mathfrak{M}_\alpha\}$ is any collection of invariant subspaces for A, then $A \oplus \sum \oplus A|_{\mathfrak{M}_\alpha}$ is reflexive if for every $T \in \mathfrak{A}_A$

there is a sequence of polynomials $\{p_n\}_1^\infty$ such that $p_n(A) \to T$ in the weak operator topology and $\|p_n(A)\| \leq K$ for all n. Using the multiplicity theory for absolutely continuous unitaries [2] and unilateral shifts [3], this observation reduces the proof to showing that $M_e \oplus M_e$ on $L^2(E) \oplus H^2$, (E a closed subset of $\{z: |z| = 1\}$), is reflexive and satisfies the boundedness condition (here $(M_e f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$ for $f \in L^2(E)$ or H^2).

Suppose $\mathrm{Lat}(M_e \oplus M_e) \subseteq \mathrm{Lat}(B)$. Then $B = B_1 \oplus B_2$ with $\mathrm{Lat}(M_e) \subseteq \mathrm{Lat}(B_2)$. Since the unilateral shift M_e on H^2 is reflexive [4], $B_2 \in \mathfrak{a}_{M_e}$. So $B_2 = M_\psi$, $\psi \in H^\infty$. By considering the $M_e \oplus M_e$ – invariant subspaces of the form

$$\mathfrak{M}_n = \left\{ (M_e^{*^n} \phi \big|_E, \phi) : \phi \in H^2 \right\}$$

we conclude that $B_1 M_e^{*^n} \phi \big|_E = M_e^{*^n} \psi \phi \big|_E$ for all $\phi \in H^2$ and $n \geq 0$. Since $\{ M_e^{*^n} \phi \big|_E : \phi \in H^2, \ n \geq 0 \}$ is dense in $L^2(E)$, $B_1 = M_{\psi|_E}$. Thus $B_1 \oplus B_2 = M_{\psi|_E} \oplus M_\psi$ for $\psi \in H^\infty$. Hence $B_1 \oplus B_2 \in \mathfrak{a}_{M_e \oplus M_e}$, and $M_e \oplus M_e$ is reflexive. Suppose $T \in \mathfrak{a}_{M_e \oplus M_e}$. Then $T = T_1 \oplus T_2$ where $T_2 \in \mathfrak{a}_{M_e}$ on H^2, so $T_2 = M_\psi$ for $\psi \in H^\infty$. Now there exist polynomials $\{p_n\}$ such that $M_{p_n} = p_n(M_e) \to M_\psi$ weakly and $\|M_{p_n}\| = \|p_n\|_\infty \leq \|\psi\|_\infty$. (See [3, Solution 33].) In fact $p_n(M_e \oplus M_e) \to T_1 \oplus T_2$ weakly and

$$\|p_n(M_e \oplus M_e)\| = \|p_n\|_\infty \leq \|\psi\|_\infty.$$

Thus the boundedness condition is satisfied.

PROOF OF THE THEOREM. Suppose $\mathrm{Lat}(V) \subseteq \mathrm{Lat}(B)$, where $V = U_s \oplus U_a \oplus U_+$. Then $B = B_1 \oplus B_2 \oplus B_3$ with $\mathrm{Lat}(U_s) \subseteq \mathrm{Lat}(B_1)$ and $\mathrm{Lat}(U_a \oplus U_+) \subseteq \mathrm{Lat}(B_2 \oplus B_3)$. Hence $B_1 \in \mathfrak{a}_{U_s}$ and $B_2 \oplus B_3 \in \mathfrak{a}_{U_a \oplus U_+}$ since U_s is unitary, hence reflexive, and $U_a \oplus U_+$ is reflexive by Lemma 3. But now $B = B_1 \oplus B_2 \oplus B_3 \in \mathfrak{a}_{U_s \oplus U_a \oplus U_+} = \mathfrak{a}_V$ by the corollary. Hence V is reflexive.

We remark that the motivation for Lemma 2 was provided by Theorem 6 in [5] which states that if U_s is a discrete (hence singular) unitary then there is a sequence of positive integers $N_k \to \infty$ such that $U_s^{N_k} \to I$ in the weak operator topology. From this it is trivial to conclude that $\mathfrak{a}_{U_s \oplus U_a} = \mathfrak{a}_{U_s} \oplus \mathfrak{a}_{U_a}$ for any absolutely continuous U_a, since $U_a^{N_k} \to 0$ in the weak operator topology.

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UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104

UNIVERSITY OF KANSAS, LAWRENCE, KANSAS 66044