KLEIN BOTTLES IN CIRCLE BUNDLES

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Abstract. We prove that the Klein bottle embeds in the total space \( E \) of an orientable \( S^1 \)-bundle over an orientable 2-manifold \( M \) if and only if \( M = S^2 \) and \( E = S^1 \times S^2 \) or the lens space \( L(4, 1) \).

In this note we apply results of [1] to generalize a result given there concerning the embedding of the Klein bottle.

Proposition. The Klein bottle embeds in the total space \( E \) of an orientable \( S^1 \)-bundle over an orientable 2-manifold \( M \) if and only if \( M = S^2 \) and \( E = S^1 \times S^2 \) or the lens space \( L(4, 1) \).

To show \( M = S^2 \) we use the following result of [1]:

Theorem [1, §4.1]. Let \( i: K \rightarrow E \) be an embedding of a nonorientable \((n-1)\)-manifold \( K \) in an orientable \( n \)-manifold \( E \). Suppose that \( \alpha \in \pi_1(K) \) reverses orientation. Then for \( \beta \in \pi_1(E) \), \( \beta^{-1}it(\alpha)\beta \in it(\pi_1(K)) \) implies \( \beta \in it(\pi_1(K)) \).

Assume \( M \neq S^2 \), so \( \pi_2(M) = 0 \). In the exact sequence of the fibration

\[ \cdots \rightarrow 0 \rightarrow \pi_1(S^1) \rightarrow \pi_1(E) \rightarrow \pi_1(M) \rightarrow 0 \]

the generator of \( \pi_1(S^1) \) is mapped to an element \( g \) in the center of \( \pi_1(E) \). (Since \( E \) is trivial over the 1-skeleton of \( M \), the inverse image of any circle in \( M \) is a torus in \( E \). Hence \( g \) commutes with a basis for \( \pi_1(E) \).) By the theorem \( g \) is in the image of \( it \). Let \( \pi_1(K) = \{ \alpha, \beta: \alpha \beta \alpha^{-1} = \beta^{-1} \} \); \( \alpha \) is the orientation reversing element. Then \( g = it(\alpha^j \beta^k) \) for some integers \( j, k \). Since \( \alpha \beta \alpha^{-1} = \beta^{-k} \), we have

\[ gi_1(\alpha^{-j+1})gi_1(\alpha^{-j-1}) = it(\alpha^j \beta^k \alpha^{-j+1} \alpha^j \beta^k \alpha^{-j-1}) = 1. \]

Therefore \( g^2 = it(\alpha^{2j}) \). \( pt(g) = 0 \) and \( \pi_1(M) \) is torsion free, so \( ptit(\alpha) = 0 \). Therefore \( it(\alpha) = g^m \) and is in the center of \( \pi_1(E) \). But then by the theorem \( it \) is onto. Thus \( ptit(\beta) \) generates \( \pi_1(M) \) which contradicts \( M \neq S^2 \).

To complete the proof of the proposition recall that the total space \( E \) of an orientable \( S^1 \)-bundle over \( S^2 \) is the lens space \( L(k, 1) \) or \( S^1 \times S^2 \) (the case \( k = 0 \)). By [1, §6] a nonorientable surface of genus \( g \) embeds...
in $L(k, 1)$ if and only if $k$ is even, $g \equiv k/2 \pmod{2}$, and $g \geq k/2$. Thus the Klein bottle, which has genus 2, embeds only in $L(4, 1)$ and $S^1 \times S^2$.

If $S^1 \times S^2$ is pictured as a family of 2-spheres parameterized by $\theta$, $0 \leq \theta < 2\pi$, then the surface swept out by a meridian rotated about the poles by $\theta/2$ is a Klein bottle.

In the $x, y$-plane let $S$ be the square with vertices at $(\pm 1, 0)$ and $(0, \pm 1)$. $L(4, 1)$ is obtained from the suspension from $(0, 0, 1)$ of $S$ in $\mathbb{R}^3$ by identifying certain points of the boundary, cf. [2, p. 223]. The surface $z = xy$ gives an embedding of the Klein bottle.

REFERENCES


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