SIGMA-AMENABLE LOCALLY COMPACT GROUPS[1]

J. W. JENKINS

ABSTRACT. Let G denote a locally compact group and $\sigma(G)$ the semigroup of nonempty compact subsets of G. The combinatorial properties of the family of all groups G for which $\sigma(G)$ is amenable is studied. The relationship between amenability of G and amenability of $\sigma(G)$ is also investigated.

1. σ-amenability. Let S be a semigroup and X a closed subspace of $m(s)$, the space of all bounded real valued functions defined of S with the sup norm. Assume that X contains the constant functions. A linear functional μ on X is a mean if μ is positive and of norm one. If for each f in X, $_sf$ is also in X, where $_sf(t) = f(st)$ for each s, t in S, and if $\mu(_sf) = \mu(f)$ for each f in X and s in S, then μ is a left invariant mean (LIM) on X.

If G is a locally compact group, G is said to be amenable (more accurately, left-amenable) if there is a LIM on the space of bounded continuous functions.

For G a locally compact group, we denote by $\sigma(G)$ the family of all nonempty compact subsets of G. $\sigma(G)$ becomes a semigroup if we define multiplication by set product, i.e. $UU' = \{uu' \mid u \in U, u' \in U'\}$ for each U, U' in $\sigma(G)$.

DEFINITION 1.1. A locally compact group G is said to be σ-amenable if there is a LIM on $m(\sigma(G))$.

For the remainder of this section we are primarily concerned with the combinatorial properties of σ-amenable groups. G and H will denote locally compact Hausdorff groups. We begin with

PROPOSITION 1.2. *If G is an abelian group or a compact group, then G is σ-amenable.*

PROOF. If G is an abelian group then quite obviously $\sigma(G)$ is commutative, and hence there is a LIM on $m(\sigma(G))$ (see Day [1]).

Suppose that G is compact. Then $G \in \sigma(G)$ and hence the point evaluation e_G, defined by $e_G(f) = f(G)$ for each f in $m(\sigma(G))$ is a mean

on $m(\sigma(G))$. Furthermore, since for each U in $\sigma(G)$ we have $UG = G$, $e_G(vf) = f(UG) = f(G) = e_G(f)$. Thus, e_G is a LIM on $m(\sigma(G))$.

PROPOSITION 1.3. *Let π be an open continuous homomorphism of G onto H, and suppose that G is σ-amenable. Then H is σ-amenable.*

PROOF. π has an obvious extension to a homomorphism $\bar{\pi}$ of $\sigma(G)$ into $\sigma(H)$. The fact that $\bar{\pi}$ is an epimorphism is an immediate consequence of π being open and continuous. The proposition then follows from Day [1].

PROPOSITION 1.4. *If H is an open subgroup of G and if G is σ-amenable then H is σ-amenable.*

PROOF. First observe that if $U \in \sigma(G)$, then there exist g_1, g_2, $\cdots$, g_n in G such that $U \subset \bigcup_{i=1}^n Hg_i$. For each f in $m(\sigma(H))$, define $\bar{f}$ in $m(\sigma(G))$ pointwise by

$$\bar{f}(U) = \frac{1}{n} \sum_{i=1}^n f(Ug_i^{-1} \cap H),$$

where $U \cap Hg \neq \emptyset \Leftrightarrow Hg = Hg_i$ for some $1 \leq i \leq n$, and $Hg_i = Hg_j \Leftrightarrow i = j$. Let μ be a LIM on $m(\sigma(G))$ and define $\bar{\mu}$ in $m^*(\sigma(H))$ by $\bar{\mu}(f) = \mu(\bar{f})$ for each f in $m(\sigma(H))$. It is easily seen that $\bar{\mu}$ is a mean on $m(\sigma(H))$. Suppose now that $V \in \sigma(H)$ and that $f \in m(\sigma(H))$. Then $\bar{\mu}(vf) = \mu(\overline{vf})$, and, for each $U \in \sigma(G)$,

$$\overline{vf}(U) = \frac{1}{n} \sum_{i=1}^n vf(Ug_i^{-1} \cap H) = \frac{1}{n} \sum_{i=1}^n f[V(Ug_i^{-1} \cap H)]$$

$$= \frac{1}{n} \sum_{i=1}^n f(VUg_i^{-1} \cap H) = v(\bar{f})(U).$$

Hence, $\bar{\mu}(vf) = \mu(v(\bar{f})) = \mu(\bar{f}) = \bar{\mu}(f)$, and $\bar{\mu}$ is a LIM on $m(\sigma(H))$.

PROPOSITION 1.5. *If G is a directed union of open σ-amenable subgroups $\{H_\alpha\}$ then G is σ-amenable.*

PROOF. If μ_α is a LIM on $m(\sigma(H_\alpha))$ then $\bar{\mu}_\alpha(f) = \mu_\alpha(f|\sigma(H))$ is a mean on $m(\sigma(G))$ that is invariant under translations by elements of $\sigma(H_\alpha)$. Since $\{H_\alpha\}$ is directed, the family $\{M_\alpha\}$, where M_α is the compact subset of means on $m(\sigma(G))$ that are $\sigma(H_\alpha)$ invariant, has the finite intersection property. Thus, since the means on $m(\sigma(G))$ is a compact set, $\bigcap M_\alpha \neq \emptyset$. Finally, since each H_α is open in G, and $G = \bigcup H_\alpha$, each U in $\sigma(G)$ is in $\sigma(H_\alpha)$ for some α. Thus, if $\mu \in \bigcap M_\alpha$, μ is a LIM on $m(\sigma(G))$.

The preceding propositions indicate that the family of σ-amenable groups has combinatorial properties similar to the family of amenable groups. However, there is one significant difference. The family of amenable groups is closed with respect to group extensions. This is not true for σ-amenable groups, as an example in §2 will show.[2] In the following proposition it is shown that certain group extensions do give rise to σ-amenable groups.

PROPOSITION 1.6. *Suppose that H is the directed union of open, compact, normal subgroups of G and that G/H is σ-amenable. Then G is σ-amenable.*

PROOF. First note that H is an open normal subgroup of G, and hence G/H is a discrete group.

Let $\{g_\alpha\}$ be a transversal of G/H, and let $H = \bigcup_{\beta \in B} H_\beta$, where each H_β is an open compact normal subgroup of G.

For each β in B and f in $m(\sigma(G))$, define f_β in $m(\sigma(G))$ by $f_\beta(U) = f(UH_\beta)$. Define $\bar{f}_\beta$ in $m(\sigma(G/H))$ by

$$\bar{f}_\beta(\{g_{\alpha_1}H, \cdots, g_{\alpha_n}H\}) = f_\beta(\{g_{\alpha_1}, \cdots, g_{\alpha_n}\}).$$

Let λ be a LIM on $m(\sigma(G/H))$, and define the mean μ_β on $m(\sigma(G))$ by $\mu_\beta(f) = \lambda(\bar{f}_\beta)$.

Let $U \in \sigma(G)$. There exist $g_{\alpha_i} \in \{G_\alpha\}$ and subsets K_i of H, for $i = 1, \cdots, n$, such that

$$U = \bigcup_{i=1}^{n} g_{\alpha_i} K_i.$$

Now,

$$\mu_\beta(Uf) = \lambda(\overline{(Uf)_\beta}),$$

and

$$\overline{(Uf)_\beta}(\{g_{\alpha'_1}H, \cdots, g_{\alpha'_m}H\}) = (Uf)_\beta(\{g_{\alpha'_1}, \cdots, g_{\alpha'_m}\}) = Uf\left(\bigcup_{i=1}^{m} g_{\alpha'_i}H_\beta\right)$$

$$= f\left(\bigcup_{i=1}^{m} UH_\beta g_{\alpha'_i}\right) = f\left[\bigcup_{i=1}^{m}\left(\bigcup_{j=1}^{n} g_{\alpha_j}K_j\right)H_\beta g_{\alpha'_i}\right].$$

for β sufficiently large, $K_j \subset H_\beta$ for $j = 1, 2, \cdots, m$, and

$$\overline{(Uf)_\beta}(\{g_{\alpha'_1}H, \cdots, g_{\alpha'_m}H\}) = f\left(\bigcup_{i=1}^{m}\left(\bigcup_{j=1}^{n} g_{\alpha_j}g_{\alpha'_i}H_\beta\right)\right).$$

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[2] The author wishes to thank E. Granirer for pointing out an error in a proof of this proposition.

If $U \in \sigma(G)$ and $U = \bigcup_{i=1}^{n} g_{\alpha_i} K_i$, define $\tilde{U}$ in $\sigma(G/H)$ by $\tilde{U} = \{g_{\alpha_1} H, \cdots, g_{\alpha_n} H\}$. Then,

$$\tilde{v}(\bar{f}_\beta)(\{g_{\alpha_1'} H, \cdots, g_{\alpha_m'} H\}) = \bar{f}_\beta(\tilde{U}\{g_{\alpha_1'} H, \cdots, g_{\alpha_m'} H\})$$

$$= f_\beta(\{g_{\alpha_1}, \cdots, g_{\alpha_n}\}\{g_{\alpha_1'}, \cdots, g_{\alpha_n'}\}) = f\left(\bigcup_{i=1}^{n}\left(\bigcup_{j=1}^{m} g_{\alpha_i} g_{\alpha_j'} H_\beta\right)\right).$$

Therefore, given U in $\sigma(G)$, for sufficiently large β we have

$$\mu_\beta(\tilde{v}f) = \lambda(\overline{(vf)}_\beta) = \lambda(\tilde{U}(\bar{f}_\beta)) = \lambda(\bar{f}_\beta) = \mu_\beta(f).$$

Any w^*-cluster point of $\{\mu_\beta\}$ is thus a LIM on $m(\sigma(G))$.

2. **Amenability and σ-amenability.** In this section we consider the relationships of amenable and σ-amenable groups.

Given a group G, let $\sigma_0(G)$ denote the set of all elements of $\sigma(G)$ which have nonempty interiors. We have

LEMMA 2.1. *There is a LIM on $\sigma(G)$ if, and only if, there is a LIM on $\sigma_0(G)$.*

PROOF. Merely observe that $\sigma_0(G)$ is an ideal of $\sigma(G)$. The lemma then follows from Wilde and Witz [6].

PROPOSITION 2.2. *If G is σ-amenable then G is amenable.*

PROOF. For each f in $CB(G)$, the bounded continuous function on G, define $\bar{f}$ in $m(\sigma_0(G))$ by

$$\bar{f}(U) = |U|^{-1} \int_U f(t) dt$$

for each U in $\sigma_0(G)$, where dt denotes left Haar measure on G and $|U|$ denotes the measure of U.

By Lemma 2.1, there is a LIM μ on $m(\sigma_0(G))$. Define $\bar{\mu}$ in $CB(G)^*$ by $\bar{\mu}(f) = \mu(\bar{f})$ for each f in $CB(G)$. It is obvious that $\bar{\mu}$ is a mean. If $g \in G$,

$$\bar{\mu}(_g f) = \mu(_g \bar{f}),$$

and

$$_g\bar{f}(U) = |U|^{-1} \int_U f(gt) dt = |U|^{-1} \int_{_gU} f(t) dt = _g(\bar{f})(U)$$

for each U in $\sigma_0(G)$. Hence

$$\bar{\mu}(_g f) = \mu(_g(\bar{f})) = \mu(\bar{f}) = \bar{\mu}(f)$$

for each f in $CB(G)$, and thus G is amenable.

PROPOSITION 2.3. *Let $G = Z_2 * Z_2$, the free product of Z_2 with Z_2. Then, G is not σ-amenable.*

PROOF. G is isomorphic to the discrete group generated by a and b where $a^2 = b^2 = e$. We show that G is not σ-amenable by producing disjoint right ideals in $\sigma(G)$.

Let $U = \{a, ab\}$ and $V = \{b, ba\}$. Then $U\sigma(G) \cap V\sigma(G) = \varnothing$. To see this consider the following: each element g in G can be written in the canonical form $g = a^{\epsilon_1}b^{\epsilon_2} \cdots a^{\epsilon_{n-1}}b^{\epsilon_n}$, where $\epsilon_i = 1$ for $2 \leq i \leq n-1$ and $\{\epsilon_1, \epsilon_n\} \subset \{0, 1\}$. Define $\rho(g) = \sum_{i=1}^n \epsilon_i$, and for each U in $\sigma(G)$, define $\rho(U) = \max\{\rho(g) \mid g \in U\}$.

If $W \in U\sigma(G)$, and $g \in W$ such that $\rho(g) = \rho(W)$ then one can easily see that in the canonical form of g, $\epsilon_1 = 1$. If $W \in V\sigma(G)$, and g is chosen similarly, $\epsilon_1 = 0$ in the canonical form of g. Thus $U\sigma(G) \cap V\sigma(G) = \varnothing$. Since $U\sigma(G)$ and $V\sigma(G)$ are right ideals of $\sigma(G)$, the proposition follows.

REMARK. It is well known that $Z_2 * Z_2$ is amenable (cf. Dixmier [2]). We can therefore conclude from the preceding propositions that the family of σ-amenable groups is properly contained in the family of amenable groups.

Proposition 2.3 also shows that some restrictions on the subgroup H of Proposition 1.6 are necessary: $Z_2 * Z_2$ contains a normal abelian subgroup, generated by ab, of finite index. Hence, not every extension of a σ-amenable group by a σ-amenable group is σ-amenable.

There are amenable groups which have nonamenable open subsemigroups (cf. Hochster [4], Jenkins [5]). Let A' be the family of all (amenable) groups such that each open subsemigroup is amenable, and let A denote the family of all σ-amenable groups. It is natural to ask if there is a inclusion relation between A and A'. Although the answer is not known, we have the following information.

PROPOSITION 2.4. *A' is not contained in A.*

PROOF. $Z_2 * Z_2$ is an example of an element of A' that is not in A. To prove that $Z_2 * Z_2$ is in A' it suffices to show that $Z_2 * Z_2$ contains no free subsemigroups on two generators (cf. Frey [2]). Assume the contrary, and let g and h be elements of $Z_2 * Z_2$ that generate a free subsemigroup. We may assume that in canonical form $g = a^{\epsilon_1}b^{\epsilon_2} \cdots a^{\epsilon_{n-1}}b^{\epsilon_n}$ where $\epsilon_i = 1$ for $1 \leq i \leq n-1$. Note that $\epsilon_n = 1$ also, for if not, $g^2 = e$. If in canonical form $h = a^{\epsilon_1}b^{\epsilon_2} \cdots a^{\epsilon_{m-1}}b^{\epsilon_m}$ where $\epsilon_i = 1$ for $1 \leq i \leq m$, then $g^m = h^n$, a contradiction. Thus, we assume that $h = b^{\epsilon_1}a^{\epsilon_2} \cdots b^{\epsilon_{m-1}}a^{\epsilon_m}$ where $\epsilon_i = 1$ for $1 \leq i \leq m$. Then (i) if $n > m$, $hg = a^{\epsilon_1}b^{\epsilon_2} \cdots b^{\epsilon_{n-m}}$ where $\epsilon_i = 1$ for $1 \leq i \leq n-m$, (ii) if $m > n$, hg

$= b^{\epsilon_1} \cdots a^{\epsilon_{m-n}}$ where $\epsilon_i = 1$ for $1 \leq i \leq m - n$, or (iii) $hg = e$. If (i) holds then $(hg)^{(n} = g^{n-m}$, a contradiction. If (ii) holds, $(hg)^m = h^{m-n}$, a contradiction. Finally, if (iii) holds, then $hg = gh$, again a contradiction. Therefore the semigroup generated by g and h is not free, and $Z_2 * Z_2 \in A'$.

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STATE UNIVERSITY OF NEW YORK AT ALBANY, ALBANY, NEW YORK 12203