A CLASS OF HYPO-DIRICHLET ALGEBRAS

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Abstract. A method is given of constructing a new class of hypo-Dirichlet algebras of given real codimension.

1. Introduction. Let $X$ be a compact Hausdorff space and $A$ a uniform algebra on $X$, i.e., a uniformly closed subalgebra of $C(X)$, the space of continuous functions on $X$, that contains the constants and separates points on $X$. Denote the real parts of the functions in $A$ by $\text{Re } A$, the set of invertible elements of $A$ by $A^{-1}$, the set of logarithms of moduli of functions in $A$ by $\log A$. Let $C(X)$ denote the space of real continuous functions on $X$. A uniform algebra on $X$ is called a hypo-Dirichlet algebra if, in addition, there exist $f_1, \ldots, f_n$ in $A^{-1}$, such that the (real) vector space spanned by $\text{Re } A$ and $\log |f_1|, \ldots, \log |f_n|$ is dense in $C(X)$. The minimal number of such $f_i$ required shall be called the codimension of $\text{Re } A$. Hypo-Dirichlet algebras were first studied by Wermer [6], and further investigated by Ahern and Sarason [1]. The object here is to exhibit a class of examples of such algebras. The proofs of several of the lemmas in this paper are modeled after [2].

2. The algebra $A$. Let $\Gamma$ be the annulus $\{Z: 1 \leq |Z| \leq 2\}$, $\gamma_1 = \{Z: |Z| = 1\}$ and $\gamma_2 = \{Z: |Z| = 2\}$. Let $\Psi$ be a homeomorphism of $\gamma_1$ on $\gamma_2$ which is orientation-preserving and singular, i.e., maps a Borel set of one-dimensional Lebesgue measure 0 onto a set of measure $4\pi$. Let $B = \{f \in C(\Gamma): f$ is analytic in $\text{int } (\Gamma)\}$, and $A = \{f \in B: f(Z) = f(\Psi(Z))\}$ for all $Z \in \gamma_1$. Let $A_\Psi = A$ restricted to $\gamma_1$. Then $A_\Psi$ is a uniformly closed algebra of continuous functions on $\gamma_1$, which contains the constants.

Theorem. $A_\Psi$ is a hypo-Dirichlet algebra on $\gamma_1$, and $\text{Re } A_\Psi$ has codimension 1 in $C_\mathbb{R}(\gamma_1)$.

Definition 1. A (complex Borel) measure $\nu$ on $\gamma_1 \cup \gamma_2$ is odd if for each Borel set $E \subset \gamma_1$, $\nu(E) = -\nu(\Psi(E))$.

Definition 2. $H$ denotes the class of measures of the form: $g(Z) dZ$ on $\gamma_1 \cup \gamma_2$, where $g$ is any function in the $L^1$ closure of $B$ restricted to $\gamma_1 \cup \gamma_2$.

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Definition 3. $W$ is the space of measures $\mu + \nu$ with $\mu \in H$, $\nu$ odd. $\overline{W}$ is the weak * closure of $W$ in the space of measures on $\gamma_1 \cup \gamma_2$.

Definition 4. A measure $\lambda$ on $\gamma_1 \cup \gamma_2$ annihilates $A$ if $\int f \, d\lambda = 0$, all $f \in A$.

Clearly, every measure in $W$ annihilates $A$. Also, if $\lambda$ annihilates $A$, then $\lambda \in \overline{W}$.

Note. The measure $-i \cdot (dZ/Z) = d\theta$ is a real measure which annihilates $B$, and it is readily seen that the only real annihilators of $B$ are of the form: $\alpha \cdot d\theta$, $\alpha$ a real constant.

Lemma 1. If $\mu \in H$, $\nu$ odd, then $\|\nu\| \leq 16\|\mu + \nu\|$.

Proof. Let $E$ be any Borel subset of $\gamma_1$ and let $m$ represent Lebesgue measure. Then there are disjoint sets $F$ and $G$ with $E = F \cup G$, $m(F) = m(\Psi(G)) = 0$. Let $K = \|\mu + \nu\|$, then $|v(F)| = |\nu(F) + \mu(F)| \leq K$, since $\mu$ is absolutely continuous. $|v(G)| = |\nu(G)| = |\mu + \nu(\Psi(G))| \leq K$ for the same reason. Hence $\|\nu\| \leq 16 K$. q.e.d.

Lemma 2. Then $W = \overline{W}$.

Proof. $Q = \{\mu + \nu : \mu \in H$, $\nu$ odd, $\|\mu\| \leq 1$, $\|\nu\| \leq 1\}$ is compact. The Krein-Smulian theorem [4, p. 429] then implies $W = \overline{W}$. q.e.d.

Lemma 3. If $\nu$ is an odd measure, then $\nu$ is absolutely continuous with respect to arc length on $\gamma_1 \cup \gamma_2$ iff $\nu = 0$.

Proof. Suppose $\nu$ is absolutely continuous. Let $E$ be a Borel subset of $\gamma_1$. Then there are disjoint sets $F$ and $G$ with $E = F \cup G$, $m(F) = m(\Psi(G)) = 0$. Hence $\nu(F) = 0$, since $\nu$ is absolutely continuous $\nu(G) = -\nu(\Psi(G)) = 0$ for the same reason. Hence $\nu(E) = 0$. q.e.d.

Lemma 4. Every real annihilator, $\lambda$, of $A$ has the form: $\lambda = \nu + \alpha \cdot d\theta$, where $\nu$ is odd and $\alpha$ is a real scalar.

Proof. Since $W$ is weak * closed, we conclude that if $\lambda$ is a measure on $\gamma_1 \cup \gamma_2$, which annihilates $A$, then $\lambda = \mu + \nu, \mu \in H$, $\nu$ odd. Write $\mu = \mu_1 + i\mu_2, \nu = \nu_1 + i\nu_2$ with $\mu_1, \mu_2, \nu_1$, and $\nu_2$ real. If $\lambda$ is real $\mu_2 + \nu_2 = 0$. Hence $\nu_2$ is absolutely continuous, hence 0. Then $\mu = \mu_1$ and $\lambda = \mu_1 + \nu_1$. q.e.d.

It follows, in particular, that $\text{Re} \ A$ has codimension $\leq 1$ in $C_R(\gamma_1)$.

Lemma 5. $A$ separates the points of $\gamma_1$. Further, given $Z_1, Z_2$ with $1 \leq |Z_1| \leq 2, 1 \leq |Z_2| < 2$ and $Z_1 \neq Z_2$, then there exists an $f$ in $A$ such that $f(Z_1) \neq f(Z_2)$.

Proof. Let $\tau_1, \tau_2$ be two points of $\gamma_1$ and let $\delta_{\tau_1}, \delta_{\tau_2}$ be the point masses at $\tau_1, \tau_2$ respectively. Unless $A$ separates $\tau_1$ and $\tau_2$, $\delta_{\tau_1} - \delta_{\tau_2}$ would
be a real annihilating measure which is not in \( W \). Now suppose \( Z_1, Z_2 \) are interior to the annulus and \( A \) fails to separate them. Let \( \sigma_1, \sigma_2 \) be the harmonic measures for \( Z_1, Z_2 \) respectively. Then \( \sigma_1 - \sigma_2 \) would be a real annihilating measure. Hence, \( \sigma_1 - \sigma_2 = \nu + \alpha \cdot d\theta \), \( \nu \) odd. However \( \sigma_1 - \sigma_2 \) is absolutely continuous, therefore \( \sigma_1 - \sigma_2 = \alpha \cdot d\theta \), contradiction. Finally, if \( Z_1 \in \gamma_1 \) and \( Z_2 \) is interior, a similar argument applies. q.e.d.

Let \( T \) be the space obtained from the closed annulus \( 1 \leq |Z| \leq 2 \) by identifying \( Z \) and \( \Psi(Z) \) if \( Z \in \gamma_1 \). Then functions in \( A \) may be regarded as continuous functions on \( T \). Evidently \( T \) is topologically a torus since \( \Psi \) is orientation-preserving.

**Lemma 6.** The space of maximal ideals of \( A \) (\( A_\Psi \)) is homeomorphic to \( T \).

**Proof.** It must be shown that, if \( h \) is a homomorphism of \( A \) onto the complex numbers, then \( h \) is evaluation at some point of \( T \). If \( h \) is not evaluation at any point of \( T \), then for each \( Z, 1 \leq |Z| \leq 2 \), there is an \( f_Z \in A \), with \( h(f_Z) = 0, f_Z(Z) \neq 0 \). Since \( T \) is compact, we can select a finite number of functions \( f_1, \ldots, f_n \) in \( A \) such that \( h(f_i) = 0, i = 1, \ldots, n \). Let \( \Delta_i \) be open sets such that \( \bigcup_i \Delta_i = \{ Z : 1 \leq |Z| \leq 2 \} \) and \( f_i \neq 0 \) in \( \Delta_i \). Let \( \sigma \) be a representing measure for \( h \) on \( \gamma_1 \cup \gamma_2 \), i.e.,

\[
\int f \cdot d\sigma = h(f \cdot f_i) = h(f) \cdot h(f_i) = 0, i = 1, \ldots, n, f \in A.
\]

Thus \( f_i \cdot d\sigma \) annihilates \( A \), therefore \( f_i \cdot d\sigma = d\mu_i + d\nu_i, \mu_i \in H, \nu_i \) odd. Hence, \( f_i \cdot d\mu_i = f_i \cdot d\nu_i \) and \( f_i \cdot d\mu_i = f_i \cdot d\nu_i \). Since the right side is odd and the left side is absolutely continuous both sides vanish. Let \( \Phi \), denote the function in \( H \) such that \( d\mu_i = \Phi_i \cdot dZ \). Then \( f_i \cdot \Phi_i = f_i \cdot \Phi_j \) a.e. on \( \gamma_1 \cup \gamma_2 \) and so \( f_i \cdot \Phi_i = f_i \cdot \Phi_j \) also for \( 1 < |Z| < 2 \). We can therefore unambiguously define \( \Phi \) on \( 1 \leq |Z| \leq 2 \) by \( \Phi(Z) = \Phi_i(Z) \cdot (f_i(Z))^{-1} \) for \( Z \in \Delta_i \). Then \( \Phi \in H \).

We define a measure \( \nu \) on \( \gamma_1 \cup \gamma_2 \) by \( d\nu = (f_i)^{-1} \cdot d\nu_i \) on \( \gamma_1 \cup \gamma_2 \) \( \cap \Delta_i \). Then \( \nu \) is well defined and odd. Then \( f_i \cdot d\sigma = f_i \cdot \Phi_i \cdot dZ + f_i \cdot d\nu_i \). Since \( f_i \neq 0 \) on \( \Delta_i \), we deduce \( d\sigma = \Phi_i \cdot dZ + d\nu_i \). But then \( 1 = f_i \cdot d\sigma = \Phi_i \cdot dZ + f_i \cdot d\nu_i = 0 \). Contradiction. q.e.d.

**Lemma 7.** There is an \( f \in A^{-1} \) whose logarithm is not single valued on \( \Gamma \).

**Proof.** We regard \( A \) as an algebra of continuous functions on \( T \). The circle: \( |Z| = 3/2 \) gives rise to a one-cycle \( l_1 \) on \( T \). Let \( l_2 \) be another one-cycle on \( T \) so that \( l_1 \) and \( l_2 \) generate \( H_1(T, Z) \). By a theorem of Arens-Royden, [5], the quotient group \( A^{-1}/\text{exp}(A) \) is isomorphic to \( H^1(T, Z) \). If \( T \) is the torus, \( H^1(T, Z) \) is a free abelian group on two
generators. Let $g_1, g_2$ be two elements of $A^{-1}$ representing these generators. Write $g_1 = e^{h_1}, g_2 = e^{h_2},$ where $h_1, h_2$ are (multi-valued) analytic functions on $\Gamma$. Let $h_1$ have period $2\pi i$ on $l_1, h_2$ have period $2\pi i$ on $l_2$. Then $m \cdot h_1 - n \cdot h_2$ has 0 period on $l_2$. Suppose it also had 0 period on $l_1$. Then $g_1^m \cdot g_2^{-n} = e^{h}$ for some $h \in A$. This contradicts the choice of $g_1, g_2$. Hence $m \cdot h - n \cdot h$ has period 0 on $l$. Therefore $f = g_1^m \cdot g_2^{-n}$ is the desired element of $A^{-1}$. q.e.d.

**Proof of Theorem.** We must show that there is an $f \in A^{-1}$ such that $\log |f| \in$ closure $\text{Re } A$. We claim the $f$ of Lemma 7 is such a function. Define a linear functional $L$ on $C_R(\gamma_1 \cup \gamma_2)$ by $L(U) = (1/2\pi) \int_{|z|=3/2} dv$ where $v$ is the harmonic conjugate of $U$. Then $L$ is continuous and linear. $L(g) = 0$ for $g \in \text{Re } A^{-1}$, but $L(\log |f|) \neq 0$ since $\int_{|z|=3/2} (\arg f) \neq 0$. Therefore $\log |f| \in$ closure $\text{Re } A^{-1}$. q.e.d.

**Note.** By identifying $n$ circles instead of 2, in a similar manner, we can construct a hypo-Dirichlet algebra that has real part of codimension $n - 1$.

**References**


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