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QUOTIENT FULL LINEAR RINGS

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ABSTRACT. We define a ring R to be an FL (full linear) ring if R is isomorphic to the full ring of linear transformations of a left vector space over a division ring. R is QFL if its left maximal quotient ring is an FL ring. In this paper we give necessary and sufficient conditions for a ring to be a QFL ring. We also generalize some results of Chase and Faith concerning subdirect sum decompositions of rings whose left maximal quotient ring is the direct product of FL rings.

If ${}_RM$ is a left R-module, then $C({}_RM)$ will denote the set of closed submodules of M. We say that $C({}_RM)$ is *atomic* if $C({}_RM)$ contains minimal nonzero elements, called *atoms*, and each nonzero element of $C({}_RM)$ contains at least one atom. ${}_RM$ is *Q-prime* if for any two atoms I_1 and I_2 of $C({}_RM)$, there exists isomorphic submodules X_1 and X_2 of M such that X_i is essential in I_i for $i=1$ and 2. If X is a subset of M, then $l_R(X)$ will denote the left annihilator in R of X.

THEOREM 1. *R is QFL if and only if the following conditions are satisfied*:

(i) $Z({}_RR)=0$,

(ii) $C({}_RR)$ *is atomic*,

(iii) ${}_RR$ *is Q-prime*.

PROOF. Let Q be the left maximal quotient ring of R. If conditions (i)–(iii) hold, then, by [2, p. 70], $C({}_QQ)\cong C({}_RR)$ under contraction, and so $C({}_QQ)$ is atomic. Since $Z({}_RR)=0$, we have that Q is a left self-injective regular ring. Thus A is an atom of $C({}_QQ)$ if and only if A is a minimal left ideal of Q. For the sake of simplicity of notation we will refer to the atoms of $C({}_QQ)$ and $C({}_RR)$ as atoms of Q and R respectively.

Suppose A_1 and A_2 are atoms of Q. By [2, p. 70], A_1 and A_2 are injective as left R-modules. If $I_1=A_1\cap R$ and $I_2=A_2\cap R$, then I_1 and I_2 are atoms of R. Since ${}_RR$ is Q-prime, there exists isomorphic left ideals X_1 and X_2 of R such that X_i is essential in I_i for $i=1$ and 2. Clearly, X_i is essential in A_i for $i=1$ and 2. The injectivity of A_2 gives that A_1 and A_2 are isomorphic. Thus $l_R(A_1)=l_R(A_2)$.

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Let S be the sum of the atoms of Q. If $\{A_\alpha\}$ is the set of atoms of Q and $A \in \{A_\alpha\}$, then

$$l_R(S) = \bigcap_\alpha l_R(A_\alpha) = l_R(A).$$

If $l_R(S) \neq 0$, then $l_Q(S) \neq 0$. By [2, p. 71], $l_Q(S) \in C({}_QQ)$, and so there exists an atom B of Q such that $B \subseteq l_Q(S)$. But $B \subseteq S$, and so $B^2 = 0$. Since B is idempotent generated we have a contradiction. Thus $l_Q(A) \cap R = l_R(A) = l_R(S) = 0$, and so $l_Q(A) = 0$. Since A was an arbitrary atom of Q, we have that $l_Q(A) = 0$ for all atoms of Q.

If B is a nonzero left ideal of Q, then there is an atom A of Q such that $A \subseteq B$. Since $l_Q(B) \subseteq l_Q(A) = 0$, we have that Q is prime.

Let W be an atom of Q. W is a minimal left ideal of Q, and $W = Qe$ where $e^2 = e \neq 0$. Since Q is prime, we have that $V = eQ$ is a minimal right ideal of Q, and $D = eQe$ is a division ring. Thus V is a left D vector space.

Let $L = \mathrm{Hom}_D(V, V)$, and define a map $q \to \bar{q}$ from Q to L, where $x\bar{q} = xq$ for $x \in V$. This correspondence is clearly a ring homomorphism. If $q_1, q_2 \in Q$ and $\bar{q}_1 = \bar{q}_2$, then $xq_1 = xq_2$ for all $x \in V$. Thus $eQ(q_1 - q_2) = 0$, and $q_1 = q_2$ by the primeness of Q. Thus $q \to \bar{q}$ is a monomorphism of Q into L.

Let $\overline{V}$ be the image of V in L, and suppose $a \in L$ and $\bar{v} \in \overline{V}$ $(v \in V)$. If $x \in V$, then $x = ex$, $v = ev$, $va \in V$ and

$$\begin{aligned} x(\overline{va}) &= x(va) = (ex)(eva) = (exe)(va) \\ &= (exev)a = (xv)a = (x\bar{v})a = x(\bar{v}a). \end{aligned}$$

Thus $\bar{v}a = \overline{va} \in \overline{V}$, and so $\overline{V}$ is a right ideal of L. Since $\overline{V} \subseteq \overline{Q} \subseteq L$ we have that L is a left quotient ring of $\overline{Q}$. $\overline{Q}$ is left self-injective, so $\overline{Q} = L$. Thus $Q \cong L$, and so R is QFL.

Conversely, suppose R is QFL. By [2, p. 70], $Z({}_RR) = 0$ and $C({}_RR) \cong C({}_QQ)$ under contraction. Thus $C({}_RR)$ is atomic.

Suppose I_1 and I_2 are two atoms of R. Let A_1 and A_2 be atoms of Q such that $A_i \cap R = I_i$ for $i = 1$ and 2. Thus A_1 and A_2 are isomorphic under some isomorphism f, and I_i is essential in A_i for $i = 1$ and 2. Let $X_1 = f^{-1}(I_2) \cap I_1$ and $X_2 = f(I_1) \cap I_2$. Clearly $X_1 \cong X_2$ under the restriction of f to X_1, and X_i is essential in I_i for $i = 1$ and 2. Thus R is Q-prime.

The following theorem is essentially a restatement of a theorem of Chase and Faith [1, Theorem 1.12].

THEOREM 2. *Let Q be the left maximal quotient ring of R. Q is a direct product of FL rings if and only if*:

(i) $Z({}_RR)=0$,
(ii) $C({}_RR)$ *is atomic.*

PROOF. If (i) and (ii) hold, then $Z({}_QQ)=0$, $Q\cong \mathrm{Hom}_R(Q, Q)$, and $C({}_RR)\cong C({}_QQ)$ under contraction. Thus $C({}_QQ)$ is atomic. By [**2**, p. 70], $C({}_QQ)$ consists of the direct summands of ${}_RQ$, and so each direct summand of ${}_RQ$ contains a minimal direct summand. By [**1**, Theorem 1.12], Q is a direct product of FL rings.

Conversely, if Q is a direct product of FL rings, then Q is regular. Thus $Z({}_RR)=0$, and $C({}_QQ)$ consists of the direct summands of ${}_RQ$. Thus $C({}_QQ)$ is atomic, and since $C({}_RR)\cong C({}_QQ)$ under contraction we have that $C({}_RR)$ is atomic.

If R is a subdirect sum of rings $\{R_\alpha \mid \alpha\in A\}$ and $S=\prod_\alpha R_\alpha$, then the subdirect sum is *essential* if R (identifying R and its canonical isomorphic image in S) is an essential left R-submodule of S. Some elementary properties of essential subdirect sums appear in [**3**].

THEOREM 3. *Let Q be the left maximal quotient ring of a ring R. Q is the direct product of FL rings if and only if R is an essential subdirect sum of QFL rings.*

PROOF. Suppose $Q=\prod_\alpha Q_\alpha$, where each Q_α is an FL ring with identity e_α. For each $\alpha\in A$, let $R_\alpha=Re_\alpha$. Suppose $\alpha\in A$ and $0\neq x_\alpha\in Q_\alpha$. Since $x_\alpha\in Q$ there is an $r\in R$ such that $0\neq rx_\alpha\in R$. If $r_\alpha=re_\alpha$, then

$$0 \neq rx_\alpha = re_\alpha x_\alpha = r_\alpha x_\alpha \in R_\alpha x_\alpha \cap R_\alpha.$$

Thus each Q_α is a left quotient ring of the corresponding R_α. Since Q_α is regular, we have that $Z({}_{R_\alpha}R_\alpha)=0$, and Q_α is the left maximal quotient ring of R_α. Thus each R_α is a QFL ring.

Clearly R is a subdirect sum of the rings $R_\alpha(r\to\{re_\alpha\})$. Since $R\subseteq\prod_\alpha R_\alpha\subseteq Q$, we have that the subdirect sum is essential.

Conversely suppose R is an essential subdirect sum of QFL rings $\{R_\alpha \mid \alpha\in A\}$. Let Q_α be the left maximal quotient ring of R_α and let $S=\prod_\alpha Q_\alpha$. Clearly $R\subseteq\prod_\alpha R_\alpha\subseteq S$.

If x is a nonzero element of S, then for some $\alpha\in A$ we have $0\neq e_\alpha x\in Q_\alpha$, where e_α is the identity of Q_α. Since Q_α is a left quotient ring of R_α it follows that

$$0 \neq R_\alpha e_\alpha x \cap R_\alpha = R_\alpha x \cap R_\alpha.$$

Since $R_\alpha x\cap R_\alpha$ is a nonzero R-submodule of $\prod_\alpha R_\alpha$, we have that $R_\alpha x\cap R_\alpha\cap R\neq 0$. Since $R_\alpha\cap R$ is an essential R_α (and R) submodule of R_α, and since $Z({}_{R_\alpha}R_\alpha)=0$, we have

$$(R_\alpha \cap R)(R_\alpha x \cap R_\alpha \cap R) \neq 0.$$

Thus

$$0 \neq (R_\alpha \cap R)(R_\alpha x \cap R_\alpha \cap R) \subseteq Rx \cap R,$$

and so S is a left quotient ring of R. Since S is regular, we have $Z({}_RR)=0$. Thus $S=Q$, and Q is the direct product of FL rings. Note that $R_\alpha = R_\alpha e_\alpha = re_\alpha$, and so the QFL components of R are uniquely determined.

References

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