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A CLASS OF EMBEDDINGS OF S^{n-1} AND B^n IN R^n

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ABSTRACT. We show that if D is an n or $(n-1)$-cell in R^n, $n>4$, and E is an $(n-2)$-cell in Bd D, with $D-E$ locally flat in R^n and E locally flat in each of Bd D and R^n, then D is locally flat in R^n.

In establishing criteria for detecting local flatness of submanifolds, a central role has been played by the following problem [**1**], [**2**].

$\gamma(n, m, k)$: If D is an m-cell in R^n, E is a k-cell in Bd D, and if $D-E$ is locally flat in R^n and E is locally flat in both R^n and Bd D, then D is locally flat in R^n.

$\gamma(n, n, n-2)$ and $\gamma(n, n-1, n-2)$, $n>3$, are the only unresolved γ-statements, and, for fixed n, these are known to be equivalent [**1**]. In this paper we show that $\gamma(n, n, n-2)$ is true for $n>4$. Of equal importance is the illustration of the utility of the 1-SS property introduced in [**3**].

DEFINITION. Let $X\subset Y$ be topological spaces. Then $Y-X$ is said to be 1-SS (1-short shrink) at $x\in X$ if for every neighborhood U of x there is a neighborhood $V\subset U$ of x such that every loop in $V-X$ which is null-homotopic in $Y-X$ is also null-homotopic in $U-X$.

THEOREM 1. *Suppose that $\Sigma^{n-1}\subset S^n$ is an $(n-1)$-sphere and that $D^{n-2}\subset\Sigma^{n-1}$ is an $(n-2)$-cell. If $\Sigma^{n-1}-D^{n-2}$ and D^{n-2} are locally flat in S^n, and D^{n-2} is locally flat in Σ^{n-1} then $S^n-\Sigma^{n-1}$ is* 1-LC *at each point of Σ^{n-1}.*

Before proving the theorem we will establish the following lemma.

LEMMA 1. *Let $\Sigma^{n-2}\subset S^n$ be an $(n-2)$-sphere which is locally flat at a point x. Then, $S^n-\Sigma^{n-2}$ is* 1-SS *at x.*

PROOF. Let U be any neighborhood of x and let V be a subset of U that is a flattening cell neighborhood for Σ^{n-2} at x, i.e., $(V, V\cap\Sigma^{n-2})\approx(I^n, I^{n-2})$. Let l be a loop in $V-\Sigma$ which is null-homotopic in $S^n-\Sigma$. By pushing radially away from x we see that l is homotopic in $V-\Sigma$ to a loop l' in Bd $V-\Sigma$ which is null-homotopic in $S^n-(\text{Int } V\cup\Sigma)$. The proof will be complete if we can show that l' is null-homotopic in

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Bd $V-\Sigma$. Since we know that l' is null-homotopic in $S^n-(\text{Int } V\cup\Sigma)$, it will suffice to show that the injection

$$\pi_1(\text{Bd } V - \Sigma) \to \pi_1(S^n - (\text{Int } V \cup \Sigma))$$

is a monomorphism. In order to do this consider the following portion of the Mayer-Vietoris sequence

$$\cdots \to H_2(S^n-(\Sigma-\text{Int } V)) \to H_1(\text{Bd } V-\Sigma) \to H_1(S^n-(\text{Int } V\cup\Sigma)) \oplus H_1(V-(\text{Bd } V\cap\Sigma)) \to H_1(S^n-(\Sigma-\text{Int } V)) \to \cdots.$$

By using Alexander duality this sequence becomes

$$\cdots \to 0 \to Z \to Z \oplus 0 \to 0 \to \cdots.$$

Hence, the inclusion of Bd $V-\Sigma$ into $S^n-(\text{Int } V\cup\Sigma)$ induces an isomorphism on first homology. But now any loop l in Bd $V-\Sigma$ which is null-homotopic in $S^n-(\text{Int } V\cup\Sigma)$ is also null-homologous in $S^n-(\text{Int } V\cup\Sigma)$, consequently null-homologous in Bd $V-\Sigma$. Since $\pi_1(\text{Bd } V-\Sigma)$ is abelian, it follows that l is null-homotopic in Bd $V-\Sigma$ and so the injection $\pi_1(\text{Bd } V-\Sigma)\to\pi_1(S^n-(\text{Int } V\cup\Sigma))$ is a monomorphism as desired.

PROOF OF THEOREM 1. Clearly $S^n-\Sigma^{n-1}$ is 1-LC at each point of $\Sigma^{n-1}-D^{n-2}$, since Σ^{n-1} is locally flat at such points. Now suppose $x\in\text{Bd } D^{n-2}$. Let U be any neighborhood of x in S^n and let $V\subset U$ be a flattening neighborhood for D^{n-2} in S^n at x, i.e., $(V, V\cap D^{n-2}) \approx (E^n, E_+^{n-2})$. Without loss of generality, we may assume that $V\cap\Sigma^{n-1}\subset B^{n-1}$ where $B^{n-1}\subset U$ is a flattening open $(n-1)$-cell neighborhood of D^{n-2} in Σ^{n-1} at x. Let $l\colon\text{Bd } I^2\to V-\Sigma^{n-1}$ be any loop, and let $f\colon I^2\to V-D^{n-2}$ be an extension of l. Clearly, there is a closed $(n-1)$-cell $D_0\subset B^{n-1}-D^{n-2}$ such that $f(I^2)\cap\Sigma^{n-1}\subset D_0$. Let G denote the closure of the complementary domain of Σ^{n-1} in S^n which does not contain $l(\text{Bd } I^2)$. Let $A=f^{-1}(G)$. Then, by Tietze's extension theorem $f| A\cap f^{-1}(\Sigma^{n-1})$ can be extended to a map $f'\colon A\to D_0$. Redefine f to be f' on A. By using a collar of D_0 in $\text{Cl}(S^n-G)$ (which exists since D_0 is locally flat), we can "pull in" f to obtain $f_*\colon I^2\to U-\Sigma^{n-1}$ and so l is null-homotopic in $U-\Sigma^{n-1}$. Hence, $S^n-\Sigma^{n-1}$ is 1-LC at x.

Suppose that $x\in\text{Int } D^{n-2}$. Since D^{n-2} is locally flat in Σ^{n-1}, we may complete D^{n-2} to an $(n-2)$-sphere $\Sigma^{n-2}\subset\Sigma^{n-1}$. Let U be any neighborhood of x in S^n, let $B^{n-1}\subset U$ be a flattening $(n-1)$-cell neighborhood of Σ^{n-2} in Σ^{n-1} at x, and let $U'\subset U$ be a neighborhood of x in S^n such that $U'\cap\Sigma^{n-1}\subset B^{n-1}$. Since $S^n-\Sigma^{n-2}$ is 1-SS at x by the lemma, there is a neighborhood $V\subset U'$ of x such that every loop in $V-\Sigma^{n-2}$ which is null-homotopic in $S^n-\Sigma^{n-2}$ is also null-homotopic in $U'-\Sigma^{n-2}$. Let $l\colon\text{Bd } I^2\to V-\Sigma^{n-1}$ be any loop. Since D^{n-2} is flat in

S^n, there is a map $f: I^2 \to S^n - D^{n-2}$ which extends l. Obviously, there is a closed, locally flat $(n-1)$-cell $D_0 \subset \Sigma^{n-1} - D^{n-2}$ such that $f(I^2) \cap \Sigma^{n-1} \subset D_0$. By making an application of Tietze's extension theorem similar to the one in the preceding paragraph, we can obtain $f_*: I^2 \to S^n - \Sigma^{n-1}$ which extends l. But this means that l is null-homotopic in $S^n - \Sigma^{n-2}$ and so by our choice of V, l is null-homotopic in $U' - \Sigma^{n-2}$, i.e., there is a map $g: I^2 \to U - \Sigma^{n-2}$ which extends l. Clearly, there are two closed, locally flat $(n-1)$-cells D_+ and D_- in $B^{n-1} - \Sigma^{n-2}$ such that $g(I^2) \cap \Sigma^{n-1} \subset D_+ \cup D_-$. Let G denote the complementary domain of Σ^{n-1} in S^n which contains $l(\mathrm{Bd}\ I^2)$. Let X denote the component of $g^{-1}(G)$ which contains Bd I^2 and consider the components of $I^2 - X$. Let A_+ be the union of all those components having frontiers whose images are contained in D_+ and let A_- be the union of all those components having frontiers whose images are contained in D_-. (By unicoherence these frontiers are connected and so their images are contained in either D_+ or D_-.) Then, by Tietze's extension theorem $g | A_+ \cap g^{-1}(\Sigma^{n-1})$ can be extended to a map $g_+: A_+ \to D_+$ and $g | A_- \cap g^{-1}(\Sigma^{n-1})$ can be extended to a map $g_-: A_- \to D_-$. Redefine g to be g_+ on A_+ and g_- on A_-. By using a collar of D_+ and D_- in $\mathrm{Cl}(G \cap U)$ (which exist since D_+ and D_- are locally flat), we can "pull in" g to obtain $g_*: I^2 \to U - \Sigma^{n-1}$ and so l is null-homotopic in $U - \Sigma^{n-1}$. Hence, Σ^{n-1} is 1-LC at x as desired.

THEOREM 2. *$\gamma(n, n, n-2)$ is true.*

PROOF. Let D^n and E^{n-2}, $n > 4$, be as in the statement of $\gamma(n, n, n-2)$. It suffices to show that Bd D is locally flat. By [4], this will be the case if Bd D can be pointwise approximated by locally flat spheres and $R^n - \mathrm{Bd}\ D$ is 1-LC at each point of Bd D. The first condition follows from the fact that Bd D is collared on one side, and the second follows from Theorem 1.

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