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NONSTANDARD THEORY OF ZARISKI RINGS[1]

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ABSTRACT. Let $*R$ be an enlargement (in the sense of A. Robinson) of a Zariski ring (R, A), let μ be the monad of zero in $*R$ when R is given the A-adic topology and set R_μ equal to the quotient ring $*R/\mu$. It is shown that (R, R_μ) is a flat couple, and R_μ is Noetherian if and only if it is semilocal. Furthermore, if R is semilocal and A is the (Jacobson) radical then R_μ is semilocal, with the same number of maximal ideals and the same (Krull) dimension as R.

1. **Introduction.** Given a ring R with identity, an R-module M, and an ideal A of R such that $\bigcap_{n=0}^{\infty} A^n = (0)$ and $\bigcap_{n=0}^{\infty} A^nM = (0)$, we may make R (resp. M) into a topological ring (R-module) by adopting $\{A^n \mid n=1, 2, \cdots\}$ (resp. $\{A^nM \mid n=1, 2, \cdots\}$) as a fundamental system of neighborhoods of zero. This topology is referred to as the A-adic topology; it is a metric topology and we shall denote the completions of R and M with respect to its A-adic topology by R^* and M^* respectively. If R is Noetherian and each of its ideals is closed in its A-adic topology, then R is called a Zariski ring with respect to A and we shall write (R, A) is a Zariski ring. If R is Noetherian, then (R, A) is a Zariski ring if and only if A is contained in its (Jacobson) radical. Noetherian rings having only a finite number of maximal ideals are called semilocal rings. We shall write $(R; P_1, \cdots, P_k)$ is a semilocal ring when R is a semilocal ring and $P_1, \cdots, P_k$ are its maximal ideals.

In this paper we investigate an extension R_μ of R whose construction is familiar to those acquainted with nonstandard analysis and its application to topology and algebra (e.g. [3], [4], [6], [7], [8], [9]). More specifically, $R_\mu = *R/\mu$ where μ is the monad of zero when R is given the A-adic topology. Both R and R^* are contained in R_μ, so as a guide to this study we look for analogues to theorems relating R to R^*. Two of the deepest of these are that if (R, A) is a Zariski

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ring, then (R^*, AR^*) is a Zariski ring and (R, R^*) is a flat couple. (This definition is due to J. P. Serre [2, p. 84]; we mean here that R and R^* are rings with a common identity, R^* contains R and is a flat R-module, and $BR^* \cap R = B$ for every ideal B of R.) These results are sufficient to prove that the (Krull) dimensions of R and R^* are equal and a number of "lying over" type relationships between the ideal structures of R and R^*. In the next section, after presenting necessary preliminary material, we prove that if (R, A) is a Zariski ring, then (R, R_μ) is a flat couple. In general R_μ is non-Noetherian and for this reason (R_μ, AR_μ) may fail to be a Zariski ring. So in the last section we look for conditions on R and A which will make R_μ Noetherian. The main result of this section is that if A is the radical of R, then R_μ is Noetherian if and only if R is semilocal.

All rings considered are commutative rings with identity and all modules are unital. The set of natural numbers will be denoted by I.

2. **Preliminaries; a flat couple.** We will assume throughout this section that (R, A) is a Zariski ring and M is a finitely generated R-module. Recall then that $\bigcap_{n \in I} A^n = (0)$ and $\bigcap_{n \in I} A^n M = (0)$ [12, p. 262] and M is Noetherian [11, p. 158].

Suppose $\mathfrak{U}$ is a higher order structure which includes I, R, M and any other R-modules relevant to the discussion. Let ${}^*\mathfrak{U}$ denote an arbitrary but fixed enlargement of $\mathfrak{U}$. (For the definition of enlargement see [7, p. 819] or [3, p. 14].) ${}^*M$ is a ${}^*R$-module. If μ and ν denote the intersection of all ${}^*(A^n)$ for all finite n and the union of all ${}^*(A^n)$ for all infinite n respectively, then $\mu = \nu$ [8, p. 447]; μ is an ideal of ${}^*R$ and we set R_μ equal to the quotient ring ${}^*R/\mu$. We will denote the natural homomorphism from ${}^*R$ to R_μ by μ. (Note that μ as an ideal and as a homomorphism depends on A for its definition, but since all monads in this paper are computed from the A-adic topology on R, we will not include this in the notation.) R and R^* are contained in R_μ [8, p. 447]; furthermore $R_\mu = R^*$ if and only if R/A^n is finite for each n. (Sufficiency is proved in [8, p. 448]. If R/A^n is infinite, it can be checked that $\mu(a_w) \notin R^*$, where $\{a_i\}_{i \in I}$ is a sequence of representatives from distinct cosets of A^n in R, and w is an infinite natural number.)

In a manner similar to that in the preceding paragraph, the intersection of all ${}^*(A^n M)$ for finite n is a submodule of ${}^*M$ equal to the union of all ${}^*(A^n M)$ for infinite n. Even at the risk of confusion we shall let μ (resp. ν) denote this submodule, and set M_μ equal to the quotient module ${}^*M/\mu$. We will also denote the natural homomorphism from ${}^*M$ to M_μ by μ. The context should make it clear which μ is being used.

A very useful result which is applied several times in the paper is that if Q is an internal submodule of $^*M$, then $\bigcap_{n\in I} (Q+{}^*(A^nM)) = Q+\mu$. Half of this equality is trivial. For the other half, let $x\in Q+{}^*(A^nM)$ for all finite n. Then $x\in Q+{}^*(A^wM)$ for some infinite natural number w. But $Q+{}^*(A^wM)\subseteq Q+\nu=Q+\mu$.

It is easy to check that $\mu(a)\mu(x)=\mu(ax)$ for $a\in {}^*R$, $x\in {}^*M$, is a well-defined product that makes M_μ into an R_μ-module. If N is an R-submodule of M then $\mu({}^*N)=({}^*N+\mu)/\mu$ is an R_μ-submodule which we will denote by N_μ. Be careful to note that in this context, N_μ does not mean $^*N/\mu'$ where μ' is the intersection of all finite powers of $^*(A^nN)$. However, $({}^*N/\mu'=)N_{\mu'}\cong N_\mu$ $(=({}^*N+\mu)/\mu)$. To see this it suffices to prove that $^*N\cap\mu=\mu'$ since $({}^*N+\mu)/\mu\cong {}^*N/({}^*N\cap\mu)$; that is, we must show that

$$^*N \cap \bigcap_{n\in I} {}^*(A^nM) = \bigcap_{n\in I} {}^*(N\cap (A^nM)) = \bigcap_{n\in I} {}^*(A^nN).$$

But this latter equality follows as an application of the Artin-Rees Lemma [5, p. 9]. Similarly, if B is an ideal of R we set $B_\mu=\mu({}^*B)$.

We note that if N is a submodule of M then $N_\mu=R_\mu N$; for suppose that $x_1, \cdots, x_k$ generate N. If $\mu(x)\in N_\mu$ then $x=y+z$ where $y\in {}^*N$, $z\in\mu$ and $y=r_1x_1+\cdots+r_kx_k$ for some $r_i\in {}^*R$. Thus $\mu(x) =\mu(r_1)x_1+\cdots+\mu(r_k)x_k$ where we have identified $\mu(x_i)$ and x_i as usual. That is, $x_1, \cdots, x_k$ also generate N_μ as an R_μ-submodule. An argument similar to that just given shows that if B is an ideal of R and N a submodule of M then $BN_\mu=B_\mu N_\mu=(BN)_\mu$. Thus $A^nM_\mu=(A^n)_\mu M_\mu=(A_\mu)^nM_\mu=(A^*)^nM_\mu=(A^nM)_\mu$, so that A-adic, the A^*-adic, the A_μ-adic topologies on M_μ coincide, and each agrees with the S-topology induced on M_μ by the topology on M [6, p. 108].

Now suppose that M' is another R-module such that $\bigcap_{n\in I} A^nM'=(0)$ and that $f:M\to M'$ is an R-homomorphism. Define $f_\mu:M_\mu\to M'_{\mu'}$ $(M'_{\mu'}={}^*(M')/\mu', \mu'=\bigcap_{n\in I} {}^*(A^nM'))$ by $f_\mu(\mu(x))=\mu'({}^*f(x))$ for $x\in {}^*M$. It is straightforward to check that f_μ is a well-defined R_μ-homomorphism.

LEMMA. *If M, M', and M'' are finitely generated R-modules and $M\xrightarrow{f}M'\xrightarrow{g}M''$ is an exact sequence of R-homomorphisms then $M_\mu\xrightarrow{f_\mu}M_{\mu'}\xrightarrow{g_\mu}M''_{\mu''}$ is an exact sequence of R_μ-homomorphisms.*

PROOF. Suppose $\mu'(x)\in\operatorname{Ker} g_\mu$. Then $\mu''({}^*g(x))=0$ so that $^*g(x)\in\mu''=\bigcap_{n\in I} {}^*(A^nM'')$. Let $n\in I$ be arbitrary but fixed. $g(M')$ is a submodule of M'', hence by the Artin-Rees Lemma, there exists $m\in I$ such that $A^mM''\cap g(M')\subseteq A^ng(M')=g(A^nM')$ and therefore

$$^*(A^mM'') \cap {}^*g({}^*M') \subseteq {}^*(g(A^nM')) = {}^*g({}^*(A^nM')).$$

It follows that $^*g(x) \in {}^*g(^*(A^nM'))$ so there exists $x' \in {}^*(A^nM')$ such that $^*g(x) = {}^*g(x')$; that is, $^*g(x-x')=0$. Therefore since $^*M \xrightarrow{^*f} {}^*(M') \xrightarrow{^*g} {}^*(M'')$ is exact, $x-x' \in {}^*f(^*M)$; that is $x \in x' + {}^*f(^*M) \subseteq {}^*f(^*M) + {}^*(A^nM')$ and $\mu'(x) \in ({}^*f(^*M)+\mu')/\mu' = f_\mu(M_\mu)$. It follows that $\operatorname{Ker} g_\mu \subseteq \operatorname{Im} f_\mu$. The reverse inclusion is easy. ∎

We make $M \otimes_R R_\mu$ into an R_μ-module by defining $\mu(a)(x\otimes\mu(b)) = x\otimes\mu(ab)$ for $a, b \in {}^*R$, $x \in M$. The preceding lemma can be used to prove that $M_\mu \cong M \otimes_R R_\mu$; the proof is completely analogous to the proof in [12, pp. 265–266] that $M^* \cong M \otimes_R R^*$ so it will be omitted here.

THEOREM 1. *If (R, A) is a Zariski ring then (R, R_μ) is a flat couple.*

PROOF. R is a subring of R_μ and they share a common identity. If B is an ideal of R then it is easy to show that $BR_\mu \cap R = B$ since B is closed in R. To show that R_μ is a flat R-module, suppose that M and M' are R-modules and $f: M \to M'$ is an injective R-module homomorphism. It is sufficient to prove the result when M and M' are finitely generated. By the preceding lemma, $f_\mu: M_\mu \to M'_{\mu'}$ is an injective homomorphism. It is easily verified that

$$\begin{array}{ccc} M_\mu & \xrightarrow{f_\mu} & M'_{\mu'} \\ \varphi_M \downarrow & & \varphi_{M'} \downarrow \\ M \otimes_R R_\mu & \xrightarrow{f\otimes 1} & M' \otimes_R R_\mu \end{array}$$

is commutative, where φ_M, $\varphi_{M'}$ are isomorphisms. Hence $f\otimes 1$ is injective. ∎

It is known that flatness implies the following [2, pp. 83–84].

COROLLARY. *If (R, A) is a Zariski ring, B is an ideal of R, and L and N submodules of M, then*

(i) $(L\cap N)_\mu = (L\cap N)R_\mu = LR_\mu \cap NR_\mu = L_\mu \cap N_\mu$,

(ii) $(N:B)_\mu = (N:B)R_\mu = NR_\mu : BR_\mu = N_\mu : B_\mu$,

(iii) $(L:N)_\mu = (L:N)R_\mu = LR_\mu : NR_\mu = L_\mu : N_\mu$.

3. **A Noetherian extension.** Throughout this section A will denote any ideal of R (not necessarily Zariski) such that $\bigcap_{n\in I} A^n = (0)$, and the monad of zero in $^*R$, μ, will always be computed relative to this A; that is, $R_\mu = {}^*R/\mu$, $\mu = \bigcap_{n\in I} {}^*(A^n)$. Rad R will denote the Jacobson radical of R.

If $A^n = (0)$ for some $n \in I$ then $R_\mu = {}^*R$ and elementary properties of enlargements make it easy to show that R_μ is Noetherian if and only if R is Artinian (also if and only if R_μ is Artinian).

THEOREM 2. *If* $(R; P_1, \cdots, P_k)$ *is a semilocal ring and* $A = \text{Rad } R$, *then* $(R_\mu; (P_1)_\mu, \cdots, (P_k)_\mu)$ *is a semilocal ring.*

PROOF. For $1 \leqq i \leqq k$, P_i a maximal ideal of R means that P_i is an ideal and for every element a of R not in P_i there is an element b in R such that $ab - 1 \in P_i$. Interpreting this in the enlargement, we see that $^*P_i$ must be a maximal ideal among all ideals (internal and external) of $^*R$. Since $A \subseteq P_i$ it is easy to show that $(P_i)_\mu \neq R_\mu$ so that the $(P_i)_\mu$ are maximal ideals of R_μ.

If Q is a maximal ideal of R_μ then $\mu^{-1}(Q)$ is a maximal ideal of $^*R$. Since $P_1, \cdots, P_k$ are finitely generated the standard and nonstandard meaning of the product $^*P_1 \cdots {}^*P_k$ coincide, so we can use the symbol without ambiguity. We have

$$^*P_1 \cdots {}^*P_k \subseteq {}^*P_1 \cap \cdots \cap {}^*P_k = {}^*(P_1 \cap \cdots \cap P_k) = {}^*(\text{Rad } R).$$

But $x \in \text{Rad } R$ if and only if for every $y \in R$ there is a $z \in R$ such that $(1 - xy)z = 1$. Interpreting this in the enlargement $^*R$, we see that $^*(\text{Rad } R) = \text{Rad } {}^*R$. It follows that $^*P_1 \cdots {}^*P_k \subseteq \text{Rad } {}^*R \subseteq \mu^{-1}(Q)$ and since $\mu^{-1}(Q)$ is prime, being maximal, $^*P_i \subseteq \mu^{-1}(Q)$ for some $i = 1, \cdots, k$. But $^*P_i$ is a maximal ideal so $^*P_i = \mu^{-1}(Q)$ and it follows that $Q = (P_i)_\mu$. Hence $(P_1)_\mu, \cdots, (P_k)_\mu$ are the maximal ideals of R_μ and they are finitely generated.

We now have a ring R_μ which Nagata calls a "semilocal ring which may not be Noetherian" [5, p. 13]. To show it is Noetherian it only remains to show that every finitely generated ideal of R_μ is a closed subset of R_μ [5, p. 110]. So let Q be a finitely generated ideal of R_μ; say its generators are $\mu(a_1), \cdots, \mu(a_m)$, $a_i \in {}^*R$, and let $Q' = {}^*Ra_1 + \cdots + {}^*Ra_m$ (note that Q' is an internal ideal of $^*R$). If $\mu(x) \notin Q$ then $x \notin Q' + \mu = \bigcap_{n \in I} (Q' + {}^*(A^n))$. There is an $n \in I$ such that $x \notin Q' + {}^*(A^n)$ and therefore $x + {}^*(A^n)$ does not intersect $Q' + {}^*(A^n)$ or $Q' + \mu$. It follows that $\mu(x) + (A^n)_\mu$ does not intersect Q so that Q is closed. ■

THEOREM 3. *If* R_μ *is Noetherian then* R_μ *is semilocal.*

PROOF. If R_μ is Noetherian then A is contained in only finitely many maximal ideals of R. For suppose A is contained in infinitely many distinct maximal ideals. Then for each $n \in I$ there is a strictly increasing sequence of ideals of length n which includes A. Thus in the enlargement there is a sequence of (internal) ideals of length w, where w is an infinite natural number. The μ-image of these ideals will give a standard infinite strictly increasing sequence of ideals in R_μ, contrary to R_μ being Noetherian.

So suppose that $P_1, \cdots, P_k$ are the maximal ideals of R containing A. $(P_1)_\mu, \cdots, (P_k)_\mu$ are maximal ideals of R_μ. Suppose Q is a maximal ideal of R_μ; let $\mu(a_1), \cdots, \mu(a_m)$ be its generators and set $Q' = {}^*Ra_1 + \cdots + {}^*Ra_m$. Since Q is maximal, $Q' + \mu = \mu^{-1}(Q)$ is a maximal ideal among all ideals of ${}^*R$. But $Q' + \mu \subseteq Q' + {}^*A$ so either ${}^*A \subseteq Q' + \mu$ or $Q' + {}^*A = {}^*R$. But this latter alternative implies (since Q' and ${}^*A$ are comaximal) that for every finite n, $Q' + ({}^*A)^n = {}^*R$. Hence ${}^*R = \bigcap_{n \in I} (Q' + {}^*(A^n)) = Q' + \mu$, a contradiction. Therefore ${}^*A \subseteq Q' + \mu = \bigcap_{n \in I} (Q' + {}^*(A^n))$. Since this last intersection is not ${}^*R$, there is a finite $n \in I$ such that ${}^*A \subseteq Q' + {}^*(A^n) \neq {}^*R$. Now in R it is true that for any m elements $b_1, \cdots, b_m$ of R, if $A \subseteq Rb_1 + \cdots + Rb_m + A^n \neq R$ then each of the b_i $(1 \leqq i \leqq m)$ belong to P_j for some j, $1 \leqq j \leqq k$, (since $Rb_1 + \cdots + Rb_k + A^n \subseteq P_j$ for some j). Interpreting this sentence in ${}^*R$, we have each of the a_i in ${}^*P_j$ for some j, $1 \leqq j \leqq k$. It follows that $Q' + \mu = {}^*P_j + \mu$ and $Q = (P_i)_\mu$. Thus R_μ is a semilocal ring. ■

The preceding theorems and their proofs imply the following.

THEOREM 4. *The following are equivalent for a Noetherian ring R with $A = \text{Rad } R$.*

(i) *R is semilocal.*

(ii) *R_μ is semilocal.*

(iii) *(R_μ, AR_μ) is a Zariski ring.*

(iv) *R_μ is Noetherian.*

REMARKS. Let $(R; P_1, \cdots, P_k)$ be a semilocal ring and $A = \text{Rad } R$. Since every ideal of R_μ is finitely generated, every ideal of R_μ is the homomorphic image of an internal ideal of ${}^*R$. Thus any sentence about R and its ideals whose truth is preserved under homomorphic images will remain true for R_μ and its ideals. But Theorems 1 and 2 are sufficient to say even more about the ideal structures of R and R_μ; for example, even though the lattice of ideals of R_μ (strictly) contains the lattice of ideals of R, we can still conclude from these results that the (Krull) dimensions of R and R_μ are equal. In fact, all the results contained in [10] (which includes the above claim) continue to hold with $\hat{R}$ (the completion of R) replaced by R_μ. All the proofs in that paper, with the exception of Proposition 2 are identical. Proposition 2 can be proved exactly as Lemma 2 to Theorem 30 in [12, p. 312].

EXAMPLE. Let R be a semilocal ring and $A = \text{Rad } R$. Define $R_1 = R$, $A_1 = A$, $\mu_1 = \mu$ and for $m > 1$ define $R_m = (R_{m-1})_{\mu_{m-1}}$, $A_m = \text{Rad } R_m = (\text{Rad } R_{m-1})_{\mu_{m-1}}$, and $\mu_m = \bigcap_{n \in I} {}^*(A_m^n)$. If R/A^k is infinite for some k then we have a strictly increasing (the canonical mapping R/A^k

$\to R_\mu/(A^k)_\mu$ is one-to-one; it is onto if and only if R/A^k is finite) sequence of semilocal rings, each of the same dimension, each with the same number of maximal ideals, and (R_n, R_m) is a flat couple for all $n, m \in I$, $1 \leqq n \leqq m$.

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