TAME ARCS ON WILD CELLS

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Abstract. We prove here that, for $n \geq 5$, every cell in $E^n$ contains a tame arc and that, for product cells $B^{n-k} \times I^k \subset E^{n-k} \times E^k = E^n$, every $k$-dimensional polyhedron $P \subset B^{n-k} \times I^k$ is tame in $E^n$.

1. Introduction. Bing showed in [1] that every 2-cell in 3-dimensional Euclidean space contains a tame arc and in [2] that there is a 2-sphere that is wild but for which all subarcs are tame. We obtain here analogous results in higher dimensions ($\geq 5$). First we show that for $n \geq 5$, any subarc of any $k$-cell in $E^n$ can be approximated by subarcs tame in $E^n$. Then we show that if $C$ is any $(m-k)$-cell in $E^{n-k}$, $I^k \subset E^k$ is the $k$-fold product of the unit interval $I, m \leq n - 2$, and $n \geq 5$, then every sub $k$-cell of $C \times I^k \subset E^{n-k} \times E^k$ is tame in $E^n$. Since there are cells in this class of factored cells that are wild at every point [10] we have a generalization of Bing's example [2] to higher dimensions.

2. The approximation theorems. First we give a few definitions. Let $X \subset M$ be closed subsets of $E^n$. Let $d$ denote the usual metric on $E^n$. A homeomorphism $h$ of $M$ is an $\epsilon$-push of $(M, X)$ if there is an isotopy $h_t$ of $M$ such that $h_0 = \text{Identity}$, $h_t = h$, $d(h_t(x), x) < \epsilon$ for each $t \in I$ and each $x \in M$, and $h_t$ is the identity outside the $\epsilon$-neighborhood of $X$ in $M$ for each $t$. If $P$ is a polyhedron and $h : P \rightarrow E^n$ is an embedding we say that $h$ is tame if there is a homeomorphism $H$ of $E^n$ such that $H \cdot h$ is piecewise linear (PL).

Lemma 1. Suppose $X$ is a compact subset of $E^n$, $\text{Int } X = \emptyset$, $X$ does not locally separate $E^n$, $G$ is a compact 1-dimensional subpolyhedron of $E^n$, $n \geq 4$, and $\epsilon > 0$. Then there is an $\epsilon$-push $h$ of $(E^n, G \cap X)$ such that $h(G) \cap X = \emptyset$.

Proof. The proof is an immediate consequence of general position and Corollary 5.6 of [3].

Lemma 2. Suppose $X$ is a compact subset of $E^n$, $\text{Int } X = \emptyset$, $X$ does not locally separate $E^n$, $P$ is a 2-dimensional subpolyhedron of $E^n$, $n \geq 4$,
and \( \epsilon > 0 \). Then there is an \( \epsilon \)-push \( h \) of \((E^n, P \cap X)\) such that \( h(P) \cap X \) is totally disconnected.

**Proof.** Let \( K \) be a triangulation of \( P \) and \( \{ K_i | i = 1, 2, \ldots \} \) the sequence of \( i \)th derived barycentric subdivisions of \( K \). We shall use Lemma 1 to construct an \( \epsilon \)-push \( h \) of \((E^n, P \cap X)\) such that \( h(U \cap h_i) \cap X = \emptyset \). Clearly \( h \) then satisfies the conclusion of Lemma 2.

Let \( \epsilon_1 = \epsilon / 2 \) and apply Lemma 1 with \((X, G, \epsilon)\) replaced by \((X, |K_1|, \epsilon_1)\), obtaining an \( \epsilon_1 \)-push \( h_1 \) of \((E^n, |K_1| \cap X)\) such that \( h_1(|K_1|) \cap X = \emptyset \). Let \( \delta_1 = d(h_1(|K_1|)), X) \) and \( \eta_1 \) be some positive number chosen depending on \( h_1 \). (See [8] or Theorem 3.4 of [7].) Set \( \epsilon_2 = \min \{ \epsilon_1 / 2, \delta_1 / 2, \eta_1 \} \). As before we obtain an \( \epsilon_2 \)-push \( h_2 \) of \((E^n, h_1(|K_2|) \cap X)\) such that \( h_2 \cdot h_1(|K_2|) \cap X = \emptyset \). Set \( h = h_2 \cdot h_1 \). Continuing in this way we obtain a sequence \( \{ h_i \} \) of homeomorphisms of \( E^n \). Since \( \epsilon_{i+1} < \epsilon_i / 2 \), \( \lim_{n \to \infty} h = h \) is an \( \epsilon \)-map of \( E^n \) supported on a compact set. Because \( \epsilon_{i+1} \leq \delta_i / 2 \), \( h_i(U \cap |K_i|) \cap X = \emptyset \) and because the \( \eta_i \) are chosen sufficiently small depending on the \( h_i \), \( h \) is a homeomorphism. Thus \( h \) is the required \( \epsilon \)-push of \((E^n, X)\).

**Theorem 3.** Let \( M \) be a PL \( m \)-manifold topologically embedded in \( E^n \), \( G \) a 1-dimensional subpolyhedron of \( M \), \( G' \) a subpolyhedron of \( G \) that is tame in \( E^n \), \( n \geq 5 \), and \( m \geq 2 \). Then for each \( \epsilon > 0 \) there is an \( \epsilon \)-embedding \( \alpha : G \to M \) such that \( \alpha \) is tame in \( E^n \) and \( \alpha|G' = \text{inclusion}: G' \to E^n \).

**Proof.** It is sufficient to consider the case that \( M \) is a 2-cell, \( G \) is an arc, and \( G' = \text{end points of } G \). Using Lemma 2 we shall construct an embedding \( \alpha : G \to M \) fixed on \( G' \) such that \( E^n - \alpha(G) \) is uniformly locally 1-connected (1-ULC). By Theorem 4.2 of [4], \( \alpha \) is thus tame.

Let \( K \) be a triangulation of \( E^n \) and \( K_1, K_2, \ldots \) the sequence of \( i \)th derived barycentric subdivisions of \( K \). It follows from general position and Lemma 2 that there is an \( \epsilon / 2 \)-push \( h_1 \) of \((E^n, G \cap |K_1|)\) such that \( h_1(|K_1|) \cap M \) is totally disconnected and \( h_1(|K_1|) \cap G' = \emptyset \). Thus there is an \( \epsilon / 2 \)-embedding \( \alpha_1 : G \to M \) fixed on \( G' \) such that \( \alpha_1(G) \cap h_1(|K_1|) = \emptyset \). Now as in the proof of Lemma 2 we set \( \epsilon_1 = \epsilon / 2 \), \( \delta_1 = d(\alpha_1(G), h_1(|K_1|)), \eta_1 > 0 \) chosen depending on \( \alpha_1 \), \( \eta_1' > 0 \) chosen depending on \( h_1 \), \( \epsilon_2 = \min \{ \epsilon_1 / 2, \delta_1 / 4, \eta_1 \} \), and \( \epsilon_2' = \min \{ \epsilon_2 / 2, \delta_1 / 4, \eta_1' \} \). Then using Lemma 2 we find an \( \epsilon_2' \)-push \( h_2 \) of \((E^n, h_1(|K_2|)) \cap M \) such that \( h_2 \cdot h_1(|K_2|) \cap M \) is totally disconnected. Let \( h_2 = h_2 \cdot h_1 \). We can again find an embedding \( \alpha_2 : G \to M \) such that \( d(\alpha_2, \alpha_1(G) \cap h_2(|K_2|)) = \emptyset \), and \( \alpha_2 \) is fixed on \( G' \). Continuing in this way we construct a sequence \( \{ h_i \} \) of homeomorphisms of \( E^n \). Because \( \epsilon_{i+1} \leq \epsilon_i / 2 \), the \( \alpha_i \) converge to an \( \epsilon \)-map \( \alpha : G \to M \). The \( \eta_i \) can be picked so as to guarantee that \( \alpha \) is an
e-embedding. Similarly the \( h_i \) converge to an \( \epsilon \)-push of \((E^n, M)\). Because \( \max \{ \epsilon_{i+1}, \epsilon_{i+1} \} \leq \delta_i/2^{i+1}, \alpha(G) \cap h(U^n_{i+1} | K^n_i) = \emptyset \). Thus \( h^{-1} \cdot \alpha(G) \cap U^n_{i+1} | K^n_i = \emptyset \), and so \( E^n - h^{-1} \cdot \alpha(G) \) is 1-ULC. Therefore \( h^{-1} \cdot \alpha \) and hence \( \alpha \) is tame.

**Corollary 3.1.** Suppose \( N \) is a PL \( n \)-manifold, \( M \) is a PL \( m \)-manifold topologically embedded in \( N \), \( G \) is a 1-dimensional polyhedron, \( G' \subseteq G \) is a subpolyhedron, \( \beta: G \to M \) is an embedding such that \( \beta | G' : G' \to N \) is tame, \( n \geq 5 \), and \( m \geq 2 \). Then for each \( \epsilon > 0 \) there is an embedding \( \alpha: G \to M \) such that \( d(\alpha, \beta) < \epsilon \), \( \alpha | G' = \beta | G' \), and \( \alpha: G \to N \) is tame.

**Proof.** First take an infinite triangulation of \( G - G' \) and approximate \( \beta \) by an embedding \( \beta': G \to M \) such that \( \beta' | G' = \beta | G' \) and \( \beta' \) is locally PL on \( G - G' \). Then apply Theorem 3 to a sequence of compact subpolyhedra of \( \beta'(G - G') \). Thus we obtain an embedding \( \alpha: G \to M \) such that \( \alpha | G' = \beta | G' \) and \( \alpha | G - G' \) is locally tame in \( E^n \). Thus \( \alpha: G \to N \) is tame (Theorem 4.2 of [4]).

**Theorem 4.** Suppose \( N \) is a PL \( n \)-manifold, \( M \) is a PL \( m \)-manifold topologically embedded in \( N \), every 2-complex of \( M \) can be approximated by a 2-complex in \( M \) that is tame in \( N \), and \( 5 \leq m \leq n - 2 \). Then each \( k \)-dimensional polyhedron \( P \) topologically embedded in \( M \), \( k < m \), can be approximated in \( M \) by embeddings that are tame in \( N \).

**Proof.** It follows from [5] and either [6] or [9] that an approximation of \( P \) is tame if its complement is 1-ULC. Such an approximation is found by modifying the proof of Theorem 3. Let \( L \) be a triangulation of \( M \) and \( L_1, L_2, \ldots \) the sequence of barycentric subdivisions. Similarly let \( K_1, K_2, \ldots \) be the sequence of barycentric subdivisions of a triangulation of \( N \). Using techniques similar to those above it is possible to construct a homeomorphism \( h \) of \( N \) such that \( h(U | K^n_i) \cap M = \emptyset \) and \( h(U | K^n_i) \cap (\cup Q) = \emptyset \) where \( Q \) is a close approximation of \( | L^n_i | \) for each \( i \) that is tame in \( N \). Now for each \( i \) we can find an arc \( A_i \subseteq M \) such that \( C_i = h( | K^n_i |) \cap M \subseteq A_i \) and \( A_i - C_i \) is locally tame in \( M \). Since \( M - C_i \) is 1-ULC, \( M - A_i \) is 1-ULC and hence \( A_i \) is tame. Using \( A_i \), we can construct, for each \( i \), a homeomorphism \( f_i \) of \( M \) moving points a distance depending on \( f_{i-1} \) so that \( f_i(P) \cap C_i = \emptyset \). Thus as in the proof of Lemma 2 and Theorem 3 we can construct an \( \epsilon \)-push \( f \) of \( (M, P) \) such that \( f(P) \cap (\cup C_i) = \emptyset \). Thus \( N - f(P) \) is 1-ULC and the required approximation has been found.

It is evident that we have actually proved the following.

**Addendum to Theorem 4.** Under the hypotheses of Theorem 4 it is possible to find for each \( \epsilon > 0 \) an \( \epsilon \)-push \( f \) of \( (M, P) \) such that \( f(P) : P \to N \) is tame.
3. Subpolyhedra of factored cells. We say that an $m$-cell $C \subset E^n$ factors $k$-times if for some homeomorphism $h : E^n \to E^n$ and some $(m-k)$-cell $B \subset E^{n-k}$, $h(C) = B \times I^k \subset E^{n-k} \times E^k$ where $I^k$ is the $k$-fold product of the interval $I$ naturally embedded in $E^k$ and $B \times I^k \subset E^{n-k} \times E^k$ is the product embedding.

Theorem 5. Suppose $C$ is an $m$-cell topologically embedded in $E^n$, $C$ factors $k$-times, $n \geq 5$, and $m \leq n - 2$. Then every embedding of any compact $k$-dimensional polyhedron into $C$ is tame in $E^n$.

Proof. Let $B$ be an $(m-k)$-cell in $E^{n-k}$, $P$ a finite $k$-dimensional polyhedron topologically embedded in $B \times I^k \subset E^{n-k} \times E^k$, $n \geq 5$, and $1 \leq k < m \leq n - 2$. It follows from [5] and either [6] or [9] that $P$ is tame in $E^n$ if $E^n - P$ is 1-ULC. However, $E^n - P$ is 1-ULC if each 2-complex in $E^n$ can be homotopied off $P$ by arbitrarily small homotopies. Let $K$ be a finite 2-complex. First find a very small homotopy of $|K|$ such that for some subdivision $K'$ each 2-cell of $K'$ either projects onto a 0- or 1-simplex of $E^{n-k}$ or else lies in $E^{n-k} \times t$ for some $t \in E^k$. Since $n - k \geq (m-k)+2$ $E^{n-k} - B$ is locally 0-connected. Thus it follows that any 0- or 1-simplex in $E^{n-k}$ can be homotopied off $B$ by a small homotopy. Thus any 2-cell of $K'$ that projects onto a 0- or 1-cell of $E^{n-k}$ can be homotopied off $B \times I^k$. Let $\sigma$ be a 2-cell of $K'$, $t \in E^k$, and $\sigma \subset E^{n-k} \times t$. For $n - k \geq 4$ it follows from Lemma 2 that there is an $\varepsilon$-push $h$ of $(E^{n-k} \times t, \sigma)$ such that $h(\sigma) \cap (B \times t)$ is 0-dimensional. For $n - k = 3$ we can use the techniques of the proof of Lemma 2 to find an embedding $h : \sigma \to E^{n-k} \times t$ such that $h(\sigma) \cap (B \times t)$ is 0-dimensional and $h$ is close to the inclusion of $\sigma$ into $E^{n-k} \times t$. Let $A = h(\sigma) \cap (B \times I^k)$. $A$ is a 0-dimensional subset of $B \times t$. Let $P$ be a $k$-dimensional polyhedron topologically embedded into $B \times I^k$. Let $T \subset P$ be defined as follows: $x \in T$ if there is a neighborhood $U$ of $x$ in $P$ and a point $y \in E^{n-k}$ such that $U \subset y \times I^k$. Then $T$ is open in $P$ and $P$ is locally tame at each point of $U$ [5]. We shall construct a map $f : B \times E^k \to B \times E^k$ such that $p_1 \circ f = p_1$ where $p_1$ = projection: $B \times E^k \to B$, $f(A) \cap P \subset T$, and $d(f, \text{Id} \mid B \times E^k)$ is small. For each $(x, t) \in A \cap (P - T)$, let $\varepsilon_x > 0$ be chosen so that for some $t_x \in E^k$ with $d(t_x, t) < \varepsilon_x$ and for all $x' \in B$ with $d(x', x) < \varepsilon_x$, $(x', t_x) \in (B \times E^k) - P$. Now for some finite number of $x \in B$, the $\varepsilon_x$-neighborhoods of the $x$'s cover $p_1(A \cap (P - T))$. Since $A$ is totally disconnected it is possible to cover $p_1(A \cap (P - T))$ by closed sets $B_1, \ldots, B_k$ that are pairwise disjoint and, for each $i = 1, \ldots, k$, there is an $x_i$ such that $B_i$ lies in the $\varepsilon_{x_i}$-neighborhood of $x_i$. Define $f(x, y) = (x, y + t_{x_i} - t)$ for $x \in B_i$. Then extend $p_2 \circ f : UB_i \times E^k \to E^k$ to a map $f_2 : B \times E^k \to E^k$ such that $d(f_2, p_2) < \varepsilon$. Then extend $f$ to $B \times E^k$ by setting $f = \text{Id} \times f_2 : B \times E^k \to B \times E^k$. Then $f(A) \cap P \subset T$. Now $f$ can be extended to an $\varepsilon$-map of $E^n$ such that
$p_1 \cdot f = p_1 : E^n \to E^{n-k}$. Thus $f \cdot h(\sigma) \cap P \subseteq T$. Since $P$ is locally tame at each point of $T$ there is an approximation $g$ of $f \cdot h$ such that $g(\sigma) \cap P = \emptyset$. Thus $E^n - P$ is 1-ULC and $P$ is tame in $E^n$.

**Corollary 5.1.** Let $C \subseteq E^n$ be an $m$-cell that factors 1-time. Let $P$ be a $k$-dimensional polyhedron topologically embedded in $C$, $k \leq m \leq n-2$, and $n \geq 5$. Then for each $\epsilon > 0$ there is an $\epsilon$-push $H$ of $(C, P)$ such that $H(P)$ is tame in $C$.

**Proof.** This is actually a corollary to the proofs of Theorem 3 and Theorem 5. Let $K$ be a triangulation of $E^n$ and suppose $C = B \times I \subseteq E^{n-1} \times E^1$. Then there is an approximation $j$ of the inclusion map $i : K^2 \to E^n$ such that $j(\{ K^2 \}) \cap C$ is a 0-dimensional subset of $B \times \{ t_1, \ldots, t_p \}$ for some numbers $t_1, \ldots, t_p \in I$. Thus for any $k$-dimensional polyhedron $P \subseteq C$, there is a small homeomorphism $h$ of $C$ such that $h(P) \cap j(\{ K^2 \}) = \emptyset$. Thus we can obtain by a sequence of such steps a small homeomorphism $H$ of $C$ such that $E^n - H(P)$ is 1-ULC. Thus $H(P)$ is tame.

**Remarks.** Do Theorem 3 and Theorem 5 remain true if the hypothesis $n \geq 5$ is replaced by $n = 4$? Does Theorem 5 remain true if the hypothesis $m \leq n-2$ is replaced by $m = n-1$? More specifically take Bing’s 2-sphere $S \subseteq E^3$ [2]. Are all subarcs of $S \times I \subseteq E^4$ tame?

Theorem 5 is sharp in the sense that there are examples of cells that factor $k$-times and for which some $(k+1)$-dimensional subcell is wild.

Daverman has independently proved Theorem 3 for the case $m = 2$.

**References**


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