TAME ARCS ON WILD CELLS

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Abstract. We prove here that, for \( n \geq 5 \), every cell in \( E^n \) contains a tame arc and that, for product cells \( B^{m-k} \times I^k \subset E^{m-k} \times E^k = E^n \), every \( k \)-dimensional polyhedron \( P \subset B^{m-k} \times I^k \) is tame in \( E^n \).

1. Introduction. Bing showed in [1] that every 2-cell in 3-dimensional Euclidean space contains a tame arc and in [2] that there is a 2-sphere that is wild but for which all subarcs are tame. We obtain here analogous results in higher dimensions (\( \geq 5 \)). First we show that for \( n \geq 5 \), any subarc of any \( k \)-cell in \( E^n \) can be approximated by subarcs tame in \( E^n \). Then we show that if \( C \) is any \( (m-k) \)-cell in \( E^{n-k} \), \( I^k \subset E^k \) is the \( k \)-fold product of the unit interval \( I \), \( m \leq n - 2 \), and \( n \geq 5 \), then every sub \( k \)-cell of \( C \times I^k \subset E^{n-k} \times E^k \) is tame in \( E^n \). Since there are cells in this class of factored cells that are wild at every point [10] we have a generalization of Bing’s example [2] to higher dimensions.

2. The approximation theorems. First we give a few definitions. Let \( X \subset M \) be closed subsets of \( E^n \). Let \( d \) denote the usual metric on \( E^n \). A homeomorphism \( h \) of \( M \) is an \( \varepsilon \)-push of \( (M, X) \) if there is an isotopy \( h_t \) of \( M \) such that \( h_0 = \text{Identity} \), \( h_1 = h \), \( d(h_t(x), x) < \varepsilon \) for each \( t \in I \) and each \( x \in M \), and \( h_t \) is the identity outside the \( \varepsilon \)-neighborhood of \( X \) in \( M \) for each \( t \). If \( P \) is a polyhedron and \( h : P \to E^n \) is an embedding we say that \( h \) is tame if there is a homeomorphism \( H \) of \( E^n \) such that \( H \circ h \) is piecewise linear (PL).

Lemma 1. Suppose \( X \) is a compact subset of \( E^n \), \( \text{Int} X = \emptyset \), \( X \) does not locally separate \( E^n \), \( G \) is a compact 1-dimensional subpolyhedron of \( E^n \), \( n \geq 4 \), and \( \varepsilon > 0 \). Then there is an \( \varepsilon \)-push \( h \) of \( (E^n, G \cap X) \) such that \( h(G) \cap X = \emptyset \).

Proof. The proof is an immediate consequence of general position and Corollary 5.6 of [3].

Lemma 2. Suppose \( X \) is a compact subset of \( E^n \), \( \text{Int} X = \emptyset \), \( X \) does not locally separate \( E^n \), \( P \) is a 2-dimensional subpolyhedron of \( E^n \), \( n \geq 4 \),
and $\epsilon > 0$. Then there is an $\epsilon$-push $h$ of $(E^n, P \cap X)$ such that $h(P) \cap X$ is totally disconnected.

Proof. Let $K$ be a triangulation of $P$ and $\{K_i, i=1, 2, \ldots\}$ the sequence of $i$th derived barycentric subdivisions of $K$. We shall use Lemma 1 to construct an $\epsilon$-push $h$ of $(E^n, P \cap X)$ such that $h(U \cup h_i^q) \cap X = \emptyset$. Clearly $h$ then satisfies the conclusion of Lemma 2.

Let $\epsilon_1 = \epsilon/2$ and apply Lemma 1 with $(X, G, \epsilon)$ replaced by $(X, |K_1|, \epsilon_1)$, obtaining an $\epsilon_1$-push $h_1$ of $(E^n, K_1^1 \cap X)$ such that $h_1(K_1^1 \cap X) = \emptyset$. Let $\delta_1 = d(h_1(|K_1^1|), X)$ and $\eta_1$ be some positive number chosen depending on $h_1$. (See [8] or Theorem 3.4 of [7].) Set $\epsilon_2 = \min\{|\epsilon_1/2, \delta_1/2, \eta_1|\}$. As before we obtain an $\epsilon_2$-push $h_2$ of $(E^n, h_1|K_2^1 \cap X)$ such that $h_2 \cdot h_1(|K_2^1| \cap X) = \emptyset$. Set $h_0 = h_2 \cdot h_1$. Continuing in this way we obtain a sequence $\{h_i\}$ of homeomorphisms of $E^n$. Since $\epsilon_i+1 < \epsilon_i/2$, $\lim_i h_i = h$ is an $\epsilon$-map of $E^n$ supported on a compact set. Because $\epsilon_i+1 < \delta_i/2$, $h(U \cup |K_i^1| \cap X = \emptyset$ and because the $\eta_i$ are chosen sufficiently small depending on the $h_i$, $h$ is a homeomorphism. Thus $h$ is the required $\epsilon$-push of $(E^n, X)$.

Theorem 3. Let $M$ be a PL $m$-manifold topologically embedded in $E^n$, $G$ a 1-dimensional subpolyhedron of $M$, $G'$ a subpolyhedron of $G$ that is tame in $E^n$, $n \geq 5$, and $m \geq 2$. Then for each $\epsilon > 0$ there is an $\epsilon$-embedding $\alpha: G \rightarrow M$ such that $\alpha$ is tame in $E^n$ and $\alpha| G' = \text{inclusion} : G' \rightarrow E^n$.

Proof. It is sufficient to consider the case that $M$ is a 2-cell, $G$ is an arc, and $G' = \{\text{end points of } G\}$. Using Lemma 2 we shall construct an embedding $\alpha: G \rightarrow M$ fixed on $G'$ such that $E^n - \alpha(G)$ is uniformly locally 1-connected (1-ULC). By Theorem 4.2 of [4], $\alpha$ is thus tame.

Let $K$ be a triangulation of $E^n$ and $K_1, K_2, \ldots$ the sequence of $i$th derived barycentric subdivisions of $K$. It follows from general position and Lemma 2 that there is an $\epsilon/2$-push $h_1$ of $(E^n, G \cap |K_1^1|)$ such that $h_1(|K_1^1| \cap M$ is totally disconnected and $h_1(|K_1^1| \cap G = \emptyset$. Thus there is an $\epsilon/2$-embedding $\alpha_1: G \rightarrow M$ fixed on $G'$ such that $\alpha_1(G) \cap |K_1^1| = \emptyset$. Now as in the proof of Lemma 2 we set $\epsilon_1 = \epsilon/2$, $\delta_1 = d(\alpha_1(G), h_1(|K_1^1|))$, $\eta_1 > 0$ chosen depending on $\alpha_1$, $\eta'_1 > 0$ chosen depending on $h_1$, $\epsilon_2 = \min\{|\epsilon_1/2, \delta_1/4, \eta_1|\}$, and $\epsilon'_2 = \min\{|\epsilon_1/2, \delta_1/4, \eta'_1|\}$. Then using Lemma 2 we find an $\epsilon'_2$-push $h'_2$ of $(E^n, \alpha_1(|K_2^1|) \cap M$ such that $h'_2 \cdot h_1(|K_2^1| \cap M$ is totally disconnected. Let $h_2 = h'_2 \cdot h_1$. We can again find an embedding $\alpha_2: G \rightarrow M$ such that $d(\alpha_2, \alpha_1|G) \cap h_2(|K_2^1|) = \emptyset$, and $\alpha_2$ is fixed on $G'$. Continuing in this way we construct a sequence $\alpha_i: G \rightarrow M$ of $\epsilon_i$-embeddings and a sequence $\{h_i\}$ of homeomorphisms of $E^n$. Because $\epsilon_i+1 < \epsilon_i/2$, the $\alpha_i$ converge to an $\epsilon$-map $\alpha: G \rightarrow M$. The $\eta_i$ can be picked so as to guarantee that $\alpha$ is an
ε-embedding. Similarly the \( h_i \) converge to an ε-push of \((E^n, M)\). Because \( \max \{ \epsilon_{i+1}, \epsilon_{i+1} \} \leq \delta_i/2^{i+1}, \alpha(G) \cap h(U^n_k, K^2_i) = \emptyset \). Thus \( h \alpha(G) \cap U^n_k, K^2_i = \emptyset \), and so \( E^n - h^{-1} \alpha(G) \) is 1-ULC. Therefore \( h^{-1} \alpha \) and hence \( \alpha \) is tame.

**Corollary 3.1.** Suppose \( N \) is a PL n-manifold, \( M \) is a PL m-manifold topologically embedded in \( N \), \( G \) is a 1-dimensional polyhedron, \( G' \subseteq G \) is a subpolyhedron, \( \beta: G \to M \) is an embedding such that \( \beta|G': G' \to N \) is tame, \( n \geq 5 \), and \( m \geq 2 \). Then for each \( \epsilon > 0 \) there is an embedding \( \alpha: G \to M \) such that \( d(\alpha, \beta) < \epsilon \), \( \alpha|G' = \beta|G' \), and \( \alpha: G \to N \) is tame.

**Proof.** First take an infinite triangulation of \( G - G' \) and approximate \( \beta \) by an embedding \( \beta': G \to M \) such that \( \beta'|G' = \beta|G' \) and \( \beta' \) is locally PL on \( G - G' \). Then apply Theorem 3 to a sequence of compact subpolyhedra of \( \beta'(G - G') \). Thus we obtain an embedding \( \alpha: G \to M \) such that \( \alpha|G' = \beta|G' \) and \( \alpha|G - G' \) is locally tame in \( E^n \). Thus \( \alpha: G \to N \) is tame (Theorem 4.2 of [4]).

**Theorem 4.** Suppose \( N \) is a PL n-manifold, \( M \) is a PL m-manifold topologically embedded in \( N \), every 2-complex of \( M \) can be approximated by a 2-complex in \( M \) that is tame in \( N \), and \( 5 \leq m \leq n - 2 \). Then each \( k \)-dimensional polyhedron \( P \) topologically embedded in \( M \), \( k < m \), can be approximated in \( M \) by embeddings that are tame in \( N \).

**Proof.** It follows from [5] and either [6] or [9] that an approximation of \( P \) is tame if its complement is 1-ULC. Such an approximation is found by modifying the proof of Theorem 3. Let \( L \) be a triangulation of \( M \) and \( L_0, L_1, L_2, \ldots \) the sequence of barycentric subdivisions. Similarly let \( K_1, K_2, \ldots \) be the sequence of barycentric subdivisions of a triangulation of \( N \). Using techniques similar to those above it is possible to construct a homeomorphism \( h \) of \( N \) such that \( h(U(K^2_i)) \cap M = \emptyset \) and \( h(U(K^2_i)) \cap (U(Q_i) = \emptyset \) where \( Q_i \) is a close approximation of \( |L^2_i| \) for each \( i \) that is tame in \( N \). Now for each \( i \) we can find an arc \( A_i \subseteq M \) such that \( C_i = h(K^2_i) \cap M \subseteq A_i \) and \( A_i - C_i \) is locally tame in \( M \). Since \( M - C_i \) is 1-ULC, \( M - A_i \) is 1-ULC and hence \( A_i \) is tame. Using \( A_i \) we can construct, for each \( i \), a homeomorphism \( f_i \) of \( M \) moving points a distance depending on \( f_{i-1} \) so that \( f_i(P) \cap C_i = \emptyset \). Thus as in the proof of Lemma 2 and Theorem 3 we can construct an \( \epsilon \)-push \( f \) of \( (M, P) \) such that \( f(P) \cap (U C_i) = \emptyset \). Thus \( N - f(P) \) is 1-ULC and the required approximation has been found.

It is evident that we have actually proved the following.

**Addendum to Theorem 4.** Under the hypotheses of Theorem 4 it is possible to find for each \( \epsilon > 0 \) an \( \epsilon \)-push \( f \) of \( (M, P) \) such that \( f|P : P \to N \) is tame.
3. Subpolyhedra of factored cells. We say that an \(m\)-cell \(C \subseteq E^n\) factors \(k\)-times if for some homeomorphism \(h: E^m \rightarrow E^n\) and some \((m-k)\)-cell \(B \subseteq E^{n-k}\), \(h(C) = B \times I^k \subseteq E^{n-k} \times E^k\) where \(I^k\) is the \(k\)-fold product of the interval \(I\) naturally embedded in \(E^k\) and \(B \times I^k \subseteq E^{n-k} \times E^k\) is the product embedding.

**Theorem 5.** Suppose \(C\) is an \(m\)-cell topologically embedded in \(E^n\), \(C\) factors \(k\)-times, \(n \geq 5\), and \(m \leq n - 2\). Then every embedding of any compact \(k\)-dimensional polyhedron into \(C\) is tame in \(E^n\).

**Proof.** Let \(B\) be an \((m-k)\)-cell in \(E^{n-k}\), \(P\) a finite \(k\)-dimensional polyhedron topologically embedded in \(B \times I^k \subseteq E^{n-k} \times E^k\), \(n \geq 5\), and \(1 \leq k \leq m \leq n - 2\). It follows from \([5]\) and either \([6]\) or \([9]\) that \(P\) is tame in \(E^n\) if \(E^n - P\) is 1-ULC. However, \(E^n - P\) is 1-ULC if each 2-complex in \(E^n\) can be homotoped off \(P\) by arbitrarily small homotopies. Let \(K\) be a finite 2-complex. First find a very small homotopy of \(|K|\) such that for some subdivision \(K'\) each 2-cell of \(K'\) either projects onto a 0- or 1-simplex of \(E^{n-k}\) or else lies in \(E^{n-k} \times t\) for some \(t \in E^k\). Since \(n - k \geq (m-k) + 2E^{n-k} - B\) is locally 0-connected. Thus it follows that any 0- or 1-simplex in \(E^{n-k}\) can be homotoped off \(B\) by a small homotopy. Thus any 2-cell of \(K'\) that projects onto a 0- or 1-cell of \(E^{n-k}\) can be homotoped off \(B \times I^k\). Let \(\sigma\) be a 2-cell of \(K'\), \(t \in E^k\), and \(\sigma \subseteq E^{n-k} \times t\). For \(n - k \geq 4\) it follows from Lemma 2 that there is an \(\varepsilon\)-push \(h\) of \((E^{n-k} \times t, \sigma)\) such that \(h(\sigma) \cap (B \times t)\) is 0-dimensional. For \(n - k = 3\) we can use the techniques of the proof of Lemma 2 to find an embedding \(h: \sigma \rightarrow E^{n-k} \times t\) such that \(h(\sigma) \cap (B \times t)\) is 0-dimensional and \(h\) is close to the inclusion of \(\sigma\) into \(E^{n-k} \times t\). Let \(A = h(\sigma) \cap (B \times I^k)\). \(A\) is a 0-dimensional subset of \(B \times t\). Let \(P\) be a \(k\)-dimensional polyhedron topologically embedded into \(B \times I^k\). Let \(T \subseteq P\) be defined as follows: \(x \in T\) if there is a neighborhood \(U\) of \(x\) in \(P\) and a point \(y \in E^{n-k}\) such that \(U \subseteq y \times I^k\). Then \(T\) is open in \(P\) and \(P\) is locally tame at each point of \(U\) \([5]\). We shall construct a map \(f: B \times E^k \rightarrow B \times E^k\) such that \(p_1 \cdot f = p_1\) where \(p_1\) = projection: \(B \times E^k \rightarrow B\), \(f(A) \cap P \subseteq T\), and \(d(f, \text{Id}|B \times E^k)\) is small. For each \((x, t) \in A \cap (P - T)\), let \(\varepsilon_x > 0\) be chosen so that for some \(t_x \in E^k\) with \(d(t_x, t) < \varepsilon_x\) and for all \(x' \in B\) with \(d(x', x) < \varepsilon_x\), \((x', t_x) \in (B \times E^k) - P\). Now for some finite number of \(x \in B\), the \(\varepsilon_x\)-neighborhoods of the \(x\)'s cover \(p_1(A \cap (P - T))\). Since \(A\) is totally disconnected it is possible to cover \(p_1(A \cap (P - T))\) by closed sets \(B_1, \ldots, B_k\) that are pairwise disjoint and, for each \(i = 1, \ldots, k\), there is an \(x_i\) such that \(B_i\) lies in the \(\varepsilon_{x_i}\)-neighborhood of \(x_i\). Define \(f(x, y) = (x, y + t_{x_i} - t)\) for \(x \in B_i\). Then extend \(p_2: \bigcup B_i \times E^n \rightarrow E^n\) to a map \(f_2: B \times E^k \rightarrow E^k\) such that \(d(f_2, p_2) < \varepsilon\). Then extend \(f\) to \(B \times E^k\) by setting \(f = \text{Id} \times f_2: B \times E^k \rightarrow B \times E^k\). Then \(f(A) \cap P \subseteq T\). Now \(f\) can be extended to an \(\varepsilon\)-map of \(E^n\) such that
$p_1 \cdot f = p_1 : E^n \to E^{n-k}$. Thus $f \cdot h(\sigma) \cap P \subset T$. Since $P$ is locally tame at each point of $T$ there is an approximation $g$ of $f \cdot h$ such that $g(\sigma) \cap P = \emptyset$. Thus $E^n - P$ is 1-ULC and $P$ is tame in $E^n$.

**Corollary 5.1.** Let $C \subseteq E^n$ be an $m$-cell that factors 1-time. Let $P$ be a $k$-dimensional polyhedron topologically embedded in $C$, $k < m \leq n - 2$, and $n \geq 5$. Then for each $\varepsilon > 0$ there is an $\varepsilon$-push $H$ of $(C, P)$ such that $H(P)$ is tame in $E^n$.

**Proof.** This is actually a corollary to the proofs of Theorem 3 and Theorem 5. Let $K$ be a triangulation of $E^n$ and suppose $C = B \times I \subseteq E^{n-1} \times E^1$. Then there is an approximation $j$ of the inclusion map $i : [K^2] \to E^n$ such that $j([K^2]) \cap C$ is a 0-dimensional subset of $B \times \{t_1, \ldots, t_p\}$ for some numbers $t_1, \ldots, t_p \in I$. Thus for any $k$-dimensional polyhedron $P \subseteq C$, there is a small homeomorphism $h$ of $C$ such that $h(P) \cap j([K^2]) = \emptyset$. Thus we can obtain by a sequence of such steps a small homeomorphism $H$ of $C$ such that $E^n - H(P)$ is 1-ULC. Thus $H(P)$ is tame.

**Remarks.** Do Theorem 3 and Theorem 5 remain true if the hypothesis $n \geq 5$ is replaced by $n = 4$? Does Theorem 5 remain true if the hypothesis $m \leq n - 2$ is replaced by $m = n - 1$? More specifically take Bing’s 2-sphere $S \subseteq E^4$ tame?

Theorem 5 is sharp in the sense that there are examples of cells that factor $k$-times and for which some $(k+1)$-dimensional subcell is wild.

Davidson has independently proved Theorem 3 for the case $m = 2$.

**References**


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