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NOTE ON ALGEBRAIC LIE ALGEBRAS

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ABSTRACT. It is shown that, over an algebraically closed field of characteristic 0, the isomorphism classes of algebraic Lie algebras are in bijective correspondence with the isomorphism classes of affine algebraic groups with unipotent centers.

A Lie algebra is said to be algebraic if it is isomorphic with the Lie algebra of an affine algebraic group. In view of the fact that entirely unrelated affine algebraic groups (typically, vector groups and toroidal groups) may have isomorphic Lie algebras, this notion of *algebraic Lie algebra* calls for some clarification. The most relevant result in this direction is due to M. Goto. It says that a finite-dimensional Lie algebra L over a field of characteristic 0 is algebraic if and only if the image of L under the adjoint representation is the Lie algebra of an algebraic subgroup of the group of automorphisms of L [1, Proposition 3, p. 156]. Our main result is obtained by examining the ingredients of this characterization of algebraic Lie algebras. In particular, we shall see that, *over an algebraically closed field of characteristic* 0, *there is exactly one isomorphism class of connected affine algebraic groups with unipotent center whose Lie algebras are isomorphic with a given algebraic Lie algebra.*

The key to the group constructions we wish to make is the particularly transparent theory of *unipotent* affine algebraic groups over a field F of characteristic 0. What we shall make use of is the fact that *the category of these groups is equivalent to the category of the finite-dimensional nilpotent Lie algebras over F.* Together with the functor $\mathbf{L}$ from groups to Lie algebras, the following functor from nilpotent Lie algebras to unipotent groups implements this category equivalence. Let L be a finite-dimensional nilpotent Lie algebra over F. Let $\mathbf{U}(L)$ be the universal enveloping algebra of L, regarded as a Hopf algebra. Let $\mathbf{B}(L)$ denote F-space consisting of all those F-linear functions $\mathbf{U}(L)\rightarrow F$ which annihilate some power of the ideal $L\mathbf{U}(L)$. The Hopf algebra structure of $\mathbf{U}(L)$ dualizes into a Hopf algebra structure of $\mathbf{B}(L)$. The F-algebra structure of $\mathbf{B}(L)$ turns out to be that of an ordinary polynomial algebra. We define $\mathbf{G}(L)$ as the affine algebraic group whose elements are the F-algebra homomorphisms $\mathbf{B}(L)\rightarrow F$

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and whose group multiplication is defined in the natural way from the comultiplication of $\mathbf{B}(L)$. It is known that $\mathbf{G}(L)$ is a unipotent affine algebraic group whose algebra of polynomial functions may be identified with $\mathbf{B}(L)$ and whose Lie algebra may be identified with L. We refer the reader to [**2**, §3] for proof of these assertions. It is clear from the construction of $\mathbf{G}(L)$ that this yields a functor $\mathbf{G}$ from the category of finite-dimensional nilpotent Lie algebras over F to the category of affine algebraic groups. The pair $(\mathbf{L}, \mathbf{G})$ of functors establishes the equivalence of the category of unipotent affine algebraic groups over F with the category of finite-dimensional nilpotent Lie algebras over F (the functor isomorphisms of $\mathbf{L} \circ \mathbf{G}$ and $\mathbf{G} \circ \mathbf{L}$ with the appropriate identity functors are transparent).

Now we are in a position to construct an affine algebraic group whose Lie algebra is isomorphic with a given Lie algebra satisfying the adjoint criterion. In precise terms, let L be a finite-dimensional Lie algebra over a field F of characteristic 0, and suppose that there is a connected algebraic subgroup K of the group of automorphisms of L such that the Lie algebra $\mathbf{L}(K)$ is the Lie algebra of all inner derivations of L (i.e. the adjoint image of L). Let K_u denote the unipotent radical of K, i.e. the maximum normal unipotent subgroup of K. According to the standard structure theorem [**4**, Theorem **7.1**], there is a reductive algebraic subgroup P of K such that K is the semidirect product $K_u \cdot P$. Let $\alpha: L \to \mathbf{L}(K)$ be the adjoint representation of L, and let M denote the maximum nilpotent ideal of L. Now $\mathbf{L}(K)$ is the semidirect Lie algebra sum $\mathbf{L}(K_u)+\mathbf{L}(P)$, the natural representation of $\mathbf{L}(K_u)$ on L is nilpotent, and the natural representation of $\mathbf{L}(P)$ on L is semisimple. It is clear from this that $\alpha(M) = \mathbf{L}(K_u)$.

Let us write S for the Lie subalgebra $\alpha^{-1}(\mathbf{L}(P))$ of L, so that $L = M+S$. Clearly, S and $M \cap S$ are stable under the natural action of $\mathbf{L}(P)$ on L. From the fact that the natural representation of $\mathbf{L}(P)$ on L is semisimple, it follows that there is an $\mathbf{L}(P)$-module complement, T say, for $M \cap S$ in S. The restriction of α to T is a linear isomorphism $T \to \mathbf{L}(P)$, because the kernel of α (i.e. the center of L) is contained in M. We have $[T, T] = \alpha(T)(T) = \mathbf{L}(P)(T) \subset T$. Thus, T is a Lie subalgebra of L. From the fact that $L = M+S$, we see that L is the semidirect Lie algebra sum $M+T$.

Now consider the unipotent affine algebraic group $\mathbf{G}(M)$ that corresponds to the nilpotent Lie algebra M. Via the functor $\mathbf{G}$, the action of P by Lie algebra automorphisms of M determines an action of P by affine algebraic group automorphisms of $\mathbf{G}(M)$. We use this action of P on $\mathbf{G}(M)$ to define a semidirect product group $\mathbf{G}(M) \cdot P$. Let us write U for $\mathbf{G}(M)$, and let us denote the algebras of polynomial functions of U and P by $\mathbf{A}(U)$ and $\mathbf{A}(P)$, respectively. Now recall that the exponential map $\exp: M \rightarrow U$ is an isomorphism of affine algebraic varieties, and that, if δ is an automorphism of U and δ' is the corresponding automorphism of M, we have $\delta \circ \exp = \exp \circ \delta'$. It follows that $\mathbf{A}(U)$ is locally finite as a right P-module, this module structure being what is obtained in the natural fashion from the action of P on U (see [3, §3]). As a consequence, there is one and only one structure of affine algebraic group on $U \cdot P$ such that the algebra $\mathbf{A}(U \cdot P)$ of polynomial functions is isomorphic, via the restriction maps, with the tensor product $\mathbf{A}(U) \otimes \mathbf{A}(P)$. Equivalently, the characteristic property of this affine algebraic group structure is that U and P are algebraic subgroups of $U \cdot P$ and that the translation U-fixed part of $\mathbf{A}(U \cdot P)$ makes the factor group $(U \cdot P)/U$ into an affine algebraic group isomorphic with P. The Lie algebra of $U \cdot P$ may therefore be identified with the semidirect Lie algebra sum $\mathbf{L}(U) + \mathbf{L}(P)$, and hence with $M + T = L$. In particular, L is therefore an algebraic Lie algebra.

In the case where our base field F is algebraically closed, we can make a further adjustment. Let Z denote the center of $U \cdot P$. Then $\mathbf{L}(Z)$ is identified with the center of L, so that $\mathbf{L}(Z) \subset M = \mathbf{L}(U)$. Hence the connected component of the identity in Z is contained in U. Therefore, this component is a vector group. It follows that Z is the direct product of a vector subgroup of U (actually, $U \cap Z$) and a finite group, D say.

Our base field F being algebraically closed, the factor group $(U \cdot P)/D$ is an affine algebraic group over F such that the canonical map $U \cdot P \rightarrow (U \cdot P)/D$ is a covering of affine algebraic groups. The Lie algebra of $(U \cdot P)/D$ is still isomorphic with L, and the center of $(U \cdot P)/D$ is the unipotent image of $U \cap Z$.

We have now established the first part of the following theorem (as well as the sufficiency part of the adjoint criterion).

THEOREM. *Let L be an algebraic Lie algebra over an algebraically closed field F of characteristic* 0. *There is a connected affine algebraic F-group G such that the center of G is unipotent and the Lie algebra of G is isomorphic with L. Let G and H be connected affine algebraic F-groups with unipotent center. Then every Lie algebra isomorphism* $\mathbf{L}(G) \rightarrow \mathbf{L}(H)$ *is the differential of an affine algebraic group isomorphism* $G \rightarrow H$.

The isomorphism part of this theorem is almost trivial in the case where $\mathbf{L}(G)$ and $\mathbf{L}(H)$ are semisimple. In this case, the adjoint representations of G and H are isomorphisms identifying G and H with the connected components of the identity in the automorphism groups of $\mathbf{L}(G)$ and $\mathbf{L}(H)$, respectively. If $\rho:\mathbf{L}(G)\to\mathbf{L}(H)$ is a Lie algebra isomorphism, the required affine algebraic group isomorphism $\sigma:G\to H$ is then given by $\sigma(x)=\rho\circ x\circ\rho^{-1}$ for every element x of G.

The general case can be handled in a similar fashion after applying the following elementary lemma. In order to facilitate stating it, we make the following definition.

Let L be a finite-dimensional Lie algebra, and let M denote the maximum nilpotent ideal of L. We say that L is *regular* if the centralizer of M in L is contained in M.

LEMMA. *Let L be a finite-dimensional Lie algebra over a field of characteristic* 0. *There is one and only one direct Lie algebra decomposition $L=L_0+L_1$, where L_0 is semisimple* (*or* (0)) *and L_1 is regular.*

PROOF. Let M be the maximum nilpotent ideal of L, and let K be the centralizer of M in L. Let T denote the radical of K. From the fact that K is an ideal of L, it follows that T is contained in the radical of L, whence we have $[T, T]\subset M$. Hence $[T, [T, T]]=(0)$. Thus, T is a nilpotent ideal of L, so that we must have $T\subset M$ and $[T, K]=(0)$. Applying Levi's theorem to K, we find that K is therefore the direct Lie algebra sum $T+[K, K]$ and that $[K, K]$ is semisimple. We put $[K, K]=L_0$, and we define L_1 as the centralizer of L_0 in L. From the fact that L_0 is a semisimple ideal of L, it follows that L is the direct Lie algebra sum L_0+L_1. Evidently, the maximum nilpotent ideal of L_1 is M, and the centralizer of M in L_1 is T. Therefore, L_1 is regular.

Now let us consider any direct Lie algebra decomposition $L=U+V$, where U is semisimple or (0) and V is regular. Then M is evidently the maximum nilpotent ideal of V, and $K=U+(K\cap M)$. Hence $U=[K, K]=L_0$. Clearly, V is the centralizer of U in L, so that $V=L_1$. This completes the proof of the lemma.

Returning to the proof of the theorem, let $\mathbf{L}(G)=\mathbf{L}(G)_0+\mathbf{L}(G)_1$ and $\mathbf{L}(H)=\mathbf{L}(H)_0+\mathbf{L}(H)_1$ be the Lie algebra decompositions of the lemma. Clearly, a Lie algebra isomorphism $\rho:\mathbf{L}(G)\to\mathbf{L}(H)$ induces Lie algebra isomorphisms $\mathbf{L}(G)_0\to\mathbf{L}(H)_0$ and $\mathbf{L}(G)_1\to\mathbf{L}(H)_1$. It is clear from the definition of these components of $\mathbf{L}(G)$ and $\mathbf{L}(H)$ that they are the Lie algebras of connected normal algebraic subgroups G_0, G_1 and H_0, H_1 of G and H, respectively. Moreover, the Lie algebra of $G_0\cap G_1$ is $\mathbf{L}(G)_0\cap\mathbf{L}(G)_1=(0)$, so that $G_0\cap G_1$ is a finite central subgroup of G. Since the center of G is unipotent, we must therefore have $G_0\cap G_1=(1)$, so that G is the direct product $G_0\times G_1$. Similarly, H is the direct product $H_0\times H_1$. Since we have already disposed of the case where the Lie algebras are semisimple, this shows that it will suffice to prove the existence of the required group isomorphism $G \to H$ in the case where $\mathbf{L}(G)$ and $\mathbf{L}(H)$ are regular.

In this case, we make a standard decomposition $G = G_u \cdot P$, where G_u is the unipotent radical of G, and P is a connected reductive algebraic subgroup of G. Now $\mathbf{L}(P)$ is the direct Lie algebra sum of its center, T say, and a semisimple Lie algebra, S say. Moreover, the adjoint representation of T on $\mathbf{L}(G)$ is semisimple. The maximum nilpotent ideal M of $\mathbf{L}(G)$ is evidently of the form $\mathbf{L}(G_u) + T_1$, with $T_1 \subset T$. The adjoint representation of T_1 on $\mathbf{L}(G_u)$ is nilpotent, as well as semisimple, whence T_1 must centralize $\mathbf{L}(G_u)$. Hence T_1 lies in the center of $\mathbf{L}(G) = \mathbf{L}(G_u) + T + S$. Since the center of G is unipotent, we must therefore have $T_1 \subset \mathbf{L}(G_u)$, whence $T_1 = (0)$. Thus, $\mathbf{L}(G_u)$ is the maximum nilpotent ideal of $\mathbf{L}(G)$. Since $\mathbf{L}(G)$ is regular, it follows that the centralizer of $\mathbf{L}(G_u)$ in $\mathbf{L}(P)$ is (0), whence the centralizer of G_u in P is finite. Since this centralizer is a normal subgroup of the connected affine algebraic group P, it must lie in the center of P, and therefore in the center of G. Hence we can conclude that the centralizer of G_u in P is trivial. Therefore, the adjoint representation of P on $\mathbf{L}(G_u)$ is an isomorphism of affine algebraic groups $\alpha: P \to \alpha(P)$, where $\alpha(P)$ is a connected algebraic subgroup of the group of automorphisms of $\mathbf{L}(G_u)$, the Lie algebra of $\alpha(P)$ being the adjoint image of $\mathbf{L}(P)$ in the derivation algebra of $\mathbf{L}(G_u)$.

The given Lie algebra isomorphism $\rho: \mathbf{L}(G) \to \mathbf{L}(H)$ must send the maximum nilpotent ideal $\mathbf{L}(G_u)$ of $\mathbf{L}(G)$ onto the maximum nilpotent ideal $\mathbf{L}(H_u)$ of $\mathbf{L}(H)$. It is clear from our initial discussion of unipotent groups and nilpotent Lie algebras that there is an isomorphism of affine algebraic groups $\sigma_u: G_u \to H_u$ whose differential is the restriction of ρ to $\mathbf{L}(G_u)$.

On the other hand, $\rho(\mathbf{L}(P))$ is a maximal reductive Lie subalgebra of $\mathbf{L}(H)$, and the adjoint representation of $\rho(\mathbf{L}(P))$ on $\mathbf{L}(H_u)$ is faithful. Hence there is a connected algebraic subgroup Q of H such that $\mathbf{L}(Q) = \rho(\mathbf{L}(P))$, and the adjoint representation of Q on $\mathbf{L}(H_u)$ is semisimple and has finite kernel. This shows that Q is a reductive affine algebraic group. Since $\mathbf{L}(H) = \mathbf{L}(H_u) + \mathbf{L}(Q)$, it follows from the standard structure theorem that Q is maximal reductive in H, and that H is the semidirect product $H_u \cdot Q$. As in the case of G, we see that the adjoint representation of Q on $\mathbf{L}(H_u)$ is actually an isomorphism of affine algebraic groups $\beta: Q \to \beta(Q)$, where $\beta(Q)$ is the connected algebraic subgroup of the group of automorphisms of $\mathbf{L}(H_u)$ whose Lie algebra is the adjoint image of $\rho(\mathbf{L}(P))$ in the derivation algebra of $\mathbf{L}(H_u)$. Clearly, $\beta(Q)=\rho_u \circ \alpha(P) \circ \rho_u^{-1}$, where $\rho_u:\mathbf{L}(G_u)\to\mathbf{L}(H_u)$ is the restriction of ρ. Hence we have an isomorphism of affine algebraic groups $\sigma_r:P\to Q$, where $\sigma_r(p)=\beta^{-1}(\rho_u \circ \alpha(p) \circ \rho_u^{-1})$ for every element p of P. The required isomorphism $\sigma:G\to H$ is given by $\sigma(xp)=\sigma_u(x)\sigma_r(p)$ for every element x of G_u and every element p of P. This completes the proof of the theorem.

In particular, our result shows that, *if G is a connected affine algebraic group with unipotent center over an algebraically closed field of characteristic 0 then the natural map of the automorphism group of G into the automorphism group of $\mathbf{L}(G)$ is an isomorphism.* In this connection, it should be noted that, by [**3**, Theorem 3.2], the automorphism group of G has a natural structure of affine algebraic group, defined without any reference to $\mathbf{L}(G)$. The isomorphism just described is actually an isomorphism of affine algebraic groups.

Finally, we show that our theorem cannot be strengthened so as to admit Lie algebra homomorphisms other than isomorphisms. Let F^+ stand for the 1-dimensional vector group over F, regarded as a unipotent affine algebraic group. Let F^* stand for the multiplicative group of F, viewed as a 1-dimensional reductive affine algebraic group. Let G denote the semidirect product $F^+\cdot F^*$, where $(a, u)(a_1, u_1)=(a+ua_1, uu_1)$. Let $H=F^+$. The Lie algebra $\mathbf{L}(F^+\cdot F^*)$ has an F-basis (y, x), where $[x, y]=y$, $\mathbf{L}(F^+)=Fy$, and $\mathbf{L}(F^*)=Fx$. The center of G is trivial, while H is unipotent. Let ρ be the Lie algebra homomorphism $\mathbf{L}(G)\to\mathbf{L}(H)$, where $\rho(y)=0$ and $\rho(x)=y$. Then ρ is surjective, and the kernel of ρ is the maximum nilpotent ideal Fy of $\mathbf{L}(G)$. However, ρ is *not* the differential of a homomorphism of affine algebraic groups $\sigma:G\to H$. In fact, if σ existed, we would have $\sigma(F^*)=F^+$, which is manifestly impossible.

Let Q denote the semisimple affine algebraic group of all 2 by 2 matrices of determinant 1. The center of Q is of order 2, and we let G stand for the factor group of Q modulo its center. Let V denote the natural 2-dimensional representation space for Q. Accordingly, form the semidirect product $H=V\cdot Q$, viewing it as an affine algebraic group in the evident way. The group covering $Q\to G$ induces an isomorphism $\mathbf{L}(Q)\to\mathbf{L}(G)$. The inverse of this Lie algebra isomorphism yields an injective Lie algebra homomorphism $\rho:\mathbf{L}(G)\to\mathbf{L}(H)$. Both G and H have trivial center, and $\mathbf{L}(G)$ is semisimple. Here again, ρ cannot be the differential of a homomorphism of affine algebraic groups $\sigma:G\to H$. In fact, if σ existed, it would invert the group covering $Q\to G$, which is impossible.

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