The Szegő Infimum

Berrien Moore III

Abstract. This paper studies the Szegő infimum relative to a nonnegative, essentially bounded, operator-valued weight function and obtains a necessary and sufficient condition for this infimum to be positive.

1. Let $\mathcal{C}$ denote a separable, complex Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{C}}$, $m(\cdot)$ normalized Lebesgue measure on $[0, 2\pi)$, and $L^2_{m}$ the Hilbert space of all weakly measurable functions from the unit circle $\mathbb{T}$ to $\mathcal{C}$ having square-integrable norm. The inner product in $L^2_{m}$ is

$$
(f(\cdot), g(\cdot)) = \int_0^{2\pi} \langle f(e^{it}), g(e^{it}) \rangle_{\mathcal{C}} dm(t),
$$

The Hardy subspace of $L^2_{m}$ is denoted $H^2_{m}$. See [5] and [15]. We let $\chi(\cdot)$ denote the identity function on $\mathbb{T}$ so that the unilateral shift $S_0$ on $H^2_{m}$ is given by $S_0 f(\cdot) \rightarrow \chi(\cdot) f(\cdot)$.

The algebra of weakly measurable, essentially bounded functions from $\mathbb{T}$ to the algebra $\mathfrak{B}(\mathcal{C})$ of bounded operators on $\mathcal{C}$ will be denoted by $\mathfrak{L}(\mathfrak{B}(\mathcal{C}))$. A weakly measurable function with values in $\mathfrak{B}(\mathcal{C})$ will be called integrable if its pointwise norm, necessarily a measurable function since $\mathcal{C}$ is separable, is integrable. A weight function $w(\cdot)$ is an integrable, nonnegative (positive semidefinite) function from $\mathbb{T}$ to $\mathfrak{B}(\mathcal{C})$. If in addition, $w(e^{it})$ is invertible in $\mathfrak{B}(\mathcal{C})$ for almost all $t$, we say that $w(\cdot)$ is an invertible weight function.

If $w(\cdot)$ is a weight function, then the Szegő infimum associated with $w(\cdot)$ for a vector $c$ in $\mathfrak{C}$ is

$$
d(w(\cdot), c) = \inf_p \int_0^{2\pi} \langle w(e^{it}) [c - e^{it} p(e^{it})], c - e^{it} p(e^{it}) \rangle_{\mathfrak{B}(\mathcal{C})} dm(t),
$$

where $p(\cdot)$ runs over the class $A_{\mathfrak{C}}$ of analytic trigonometric polynomials.

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In 1920, G. Szegö [16] computed the infimum in the case where the dimension, Dim $\mathbb{C}$, of $\mathbb{C}$ is 1 and obtained
\[
d(w(\cdot), 1) = \exp \int_0^{2\pi} \log w(e^{it}) dm(t) \quad \text{if } \log w(\cdot) \in L^1(dm),
\]
\[
= 0 \quad \text{if } \log w(\cdot) \notin L^1(dm).
\]
As is well known, the condition $\log w(\cdot) \in L^1(dm)$ is equivalent to the existence of an outer function $a(\cdot)$ such that $w(\cdot) = a^*(\cdot) a(\cdot)$. For a detailed discussion, see [8, Chapter IV] and [5, Lecture III].

During the late fifties, Helson and Lowdenslager [8] and Masani and Wiener [11] found necessary and sufficient conditions for $d(w(\cdot), c) > 0$ when Dim $\mathbb{C}$ is finite. In 1961, Devinatz [2] extended the study to separable, complex Hilbert spaces $\mathbb{C}$ under the hypothesis that $w(\cdot)$ is an invertible weight function. Again there is an equivalence between $d(w(\cdot), c) > 0$ for all nonzero $c$ in $\mathbb{C}$, and the existence of a particular factorization of $w(\cdot)$. If we relax the bound from below and demand that $w(\cdot)$ be an essentially bounded weight function, then the existence of an outer factorization (see [10], [3], [15, Chapter V], and [14]) for $w(\cdot)$ yields the computation of the infimum [15, p. 224]; however, $d(w(\cdot), c)$ may equal zero nontrivially for some vector $c$ in $\mathbb{C}$ [12]. It is the purpose of this paper to present a necessary and sufficient condition for $d(w(\cdot), c) > 0$ for all nonzero $c$ in $\mathbb{C}$.

In this work our approach follows closely the spirit of Rosenblum [14], Brown and Halmos [1], and Halmos [4]. To avoid certain technical difficulties, we restrict ourselves to essentially bounded weights. As noted in the remarks this restriction can be relaxed.

2. Structure. Throughout this paper $\mathfrak{S}$ will denote a complex, separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_\mathfrak{S}$, and $Y$ a nonnegative operator in $\mathcal{B}(\mathfrak{S})$. Consider the space $\mathfrak{S}(Y^+)$ of vectors in the range, $\text{Rng } Y$, of $Y$ with the operations of vector addition and scalar multiplication inherited from $\mathfrak{S}$, and with the inner product and norm defined by $\langle f, g \rangle_{Y^+} = \langle Y^+ f, Y^+ g \rangle_\mathfrak{S}$ and $\|f\|_{Y^+} = \langle f, f \rangle_{Y^+}^{1/2}$. Here $Y^+$ is the pseudo-inverse of $Y$ ([7] and [9]), and $F$ denotes the identification map of $\mathfrak{S}(Y^+)$ into $\mathfrak{S}$ with $\text{Rng } F = \text{Rng } Y$. It quickly follows that $\mathfrak{S}(Y^+)$ is a Hilbert space, and $F : \mathfrak{S}(Y^+) \to \mathfrak{S}$ is a bounded operator. Further, let $\mathfrak{M}(Y)$ denote the vector space formed from the vectors in the closure, $\text{Cl}(\text{Rng } Y)$, of the range of $Y$. Define the linear transformation $E$ from $\mathfrak{S}$ onto $\mathfrak{M}(Y)$ as identification on $\text{Cl}(\text{Rng } Y)$ and as zero on the kernel, $\text{Ker } Y$, of $Y$. For vectors $f$ and $g$ in $\mathfrak{M}(Y)$ define an inner product and norm by $\langle f, g \rangle_Y = \langle YE f, YE g \rangle_\mathfrak{S}$ and
\[ \|f\|_Y = \langle f, f \rangle_Y^{1/2}, \] where \( E^i \) is the set theoretic inverse of \( E \). The norm completion of \( \mathcal{H}(Y) \) is a Hilbert space \( \mathcal{H}(Y) \). In this context \( E \) is continuous; \( E^+ \) will replace \( E^i \) in notation.

An isometric transformation \( U \) is defined densely from \( \mathcal{H}(Y^+) \) to \( \mathcal{H}(Y) \) by

\[ U(F^+(Y^*g)) = Eg, \quad \text{where } g \in \text{Cl}(\text{Rng } Y). \]

It can be extended to a unitary operator \( U \) from \( \mathcal{H}(Y^+) \) to \( \mathcal{H}(Y) \). An examination of \( U^*U = I \) yields the following dense description of \( U^* \):

\[ U^*f = F^+Y^*E^+f, \quad \text{where } f \in \text{Rng } E. \]

Finally, there is a simple relationship between \( E \) and \( F \), namely \( F^* = U^*E \).

3. Toeplitz operators. Let \( S \) be a unilateral shift on \( \mathcal{H} \). If \( \mathcal{C} = \text{Ker } S^* \), then \( \mathcal{C} \) is a wandering subspace; the multiplicity of \( S \) is the dimension of \( \mathcal{C} \). If \( \dim \mathcal{C} = 1 \), then \( S \) is referred to as a simple shift. Since \( \mathcal{H} = \sum_{n=0}^{\infty} \oplus \mathcal{S}/\mathcal{C} \) there exists a natural unitary map \( \Phi_\mathcal{H} \) between \( \mathcal{H} \) and \( H_\mathcal{C}^2 \) such that \( \Phi_\mathcal{H}S = S\Phi_\mathcal{H} \). Following Sz.-Nagy and Foias [15], we shall refer to \( \Phi_\mathcal{H} \) as a Fourier representation.

**Definition 3.1.** Let \( \mathcal{H} \) be a Hilbert space with unilateral shift \( S \); an operator \( T \) in \( \mathcal{B}(\mathcal{H}) \) is \( S \)-Toeplitz if \( S^*TS = T \).

By an extension of a well-known result of Brown and Halmos [1] we obtain the following:

**Theorem 3.2.** Let \( S \) be a unilateral shift of finite or infinite multiplicity on \( \mathcal{H} \); let \( \mathcal{C} = \text{Ker } S^* \). A bounded operator \( T \) on \( \mathcal{H} \) is \( S \)-Toeplitz if, and only if, \( \langle \Phi_\mathcal{H}Tf, g \rangle = P^+_+(w(\cdot)f(\cdot)) \) for some \( w(\cdot) \in L^\infty[\mathcal{B}(\mathcal{C})] \) and all \( f \in \mathcal{H} \), where \( P^+_+ \) is the orthogonal projection of \( L^2_+ \) onto \( H_\mathcal{C}^2 \).

If \( T_0f(\cdot) = P^+_+(w(\cdot)f(\cdot)) \) where \( f(\cdot) \in H_\mathcal{C}^2 \) and \( w(\cdot) \) is as above, then \( T_0 \) is \( S_0 \)-Toeplitz, and also by above \( \Phi_\mathcal{H}T = T_0\Phi_\mathcal{H} \). If \( \dim \mathcal{C} = 1 \) then \( T_0 \) is the classical Toeplitz operator on \( H^2 \).

Returning to the Hilbert spaces \( \mathcal{H}(Y) \) and \( \mathcal{H}(Y^+) \), where now \( Y \) is the nonnegative square root of a nonnegative \( S \)-Toeplitz operator \( T \) in \( \mathcal{B}(\mathcal{H}) \), we induce via \( S \) natural isometries \( S_Y \) and \( S_{Y^+} \) on \( \mathcal{H}(Y) \) and \( \mathcal{H}(Y^+) \). But, first, to define these isometries we need two lemmas. The first (though in slightly different notation) is found in Rosenblum [14, p. 142]:

**Lemma 3.3.** Let \( T \) in \( \mathcal{B}(\mathcal{H}) \) be nonnegative \( S \)-Toeplitz with \( T^{1/2} = Y \). If \( E \) denotes, as above, the natural projective embedding of \( \mathcal{H} \) into \( \mathcal{H}(Y) \), then \( \|ESf\|_Y = \|Ef\|_Y \) for all \( f \in \mathcal{H} \).
**Lemma 3.4.** Let $S$, $T$, and $Y$ be as above, then $S^*(\text{Rng } Y) \subseteq \text{Rng } Y$.

**Proof.** It is known [9] that if $A \in \mathfrak{B}(\mathfrak{H})$, then $g \in \text{Rng } A + \text{Ker } A^*$ if, and only if,

$\sup \left\{ \frac{\langle f, g \rangle_{\mathfrak{H}}}{\| A f \|_{\mathfrak{H}}}: 0 \neq f \in \text{Cl}(\text{Rng } A^*) \right\} < \infty.$

Let $g \in \text{Rng } Y$ and consider $S^*g$ in

$\sup \left\{ \frac{\langle f, S^*g \rangle_{\mathfrak{H}}}{\| Yf \|_{\mathfrak{H}}}: 0 \neq f \in \text{Cl}(\text{Rng } Y) \right\}
= \sup \left\{ \frac{\langle S f, g \rangle_{\mathfrak{H}}}{\| YSf \|_{\mathfrak{H}}}: 0 \neq f \in \text{Cl}(\text{Rng } Y) \right\}
\leq \sup \left\{ \frac{\langle k, g \rangle_{\mathfrak{H}}}{\| Yk \|_{\mathfrak{H}}}: 0 \neq k \in \text{Cl}(\text{Rng } Y) \right\} < \infty.$

Hence, $S^*g \in \text{Rng } Y + \text{Ker } Y$; but by Rosenblum [14] Cl(\text{Rng } Y) is an invariant subspace for $S^*$, thus $S^*g \in \text{Rng } Y$.

The construction of isometries is now immediate. Suppose $f \in \mathfrak{H}$; then, we densely define $S_Y$ on $\mathfrak{H}(Y)$ as $S_Y(Ef) = E(Sf)$. By 3.3, $S_Y$ is defined isometrically and has an isometric extension $S_Y$ to all of $\mathfrak{H}(Y)$. We define an isometry $S_Y^+$ on $\mathfrak{H}(Y^+)$ by $S_Y^+ = U^*S_YU$.

Of crucial importance in the next section is the particular structure of $\text{Ker } S_Y^+$. If $f$ and $g$ belong to the dense subspace $F^+(\text{Rng } T)$, then

$\langle S_Y^+f, g \rangle_{Y^+} = \langle U^*S_YUf, g \rangle_{Y^+} = \langle EST^+Ff, ET^+Fg \rangle_{Y^+}
= \langle T^+Ff, S^*TT^+Fg \rangle_{\mathfrak{H}} = \langle f, F^*S^*Fg \rangle_{Y^+}.$

By continuity, $S^+_Yf = F^*S^*Ff$ for all $f \in \mathfrak{H}(Y^+)$. If we set $\mathfrak{C}_Y = \text{Ker } S_Y^+$ and $\mathfrak{C}_{Y^+} = \text{Ker } S_Y^{*+}$, then we have $U\mathfrak{C}_{Y^+} = \mathfrak{C}_Y$ and $\mathfrak{C}_{Y^+} = F^+(\mathfrak{C} \cap \text{Rng } Y)$.

**4. Szegő infimum.** Let $w(\cdot)$ be an essentially bounded weight function with values in $\mathfrak{B}(\mathfrak{C})$, and define, as in the preceding section, the nonnegative $S_\phi$-Toeplitz operator on $H^2_\phi$ induced by $w(\cdot)$. There exist a Hilbert space $\mathfrak{H}$ and operators $S$ and $T$ in $\mathfrak{B}(\mathfrak{H})$, $S$ a unilateral shift and $T$ nonnegative $S$-Toeplitz, such that $\Phi_\phi S = S_\phi \Phi_\phi$, $\Phi_\phi T
= T_\phi \Phi_\phi$.

If $c(\cdot)$ is a constant function in $H^2_\phi$ such that $c(e^{it}) = c$ a.e. where $c$ is in $C$ and $\| c \|_\phi = 1$, then equivalent statements of the Szegő infimum are
\[
\inf_{T(t) \in \mathbb{H}^2} \int_0^{2\pi} \langle w(e^{it})[c - e^{it}f(e^{it})], c - e^{it}f(e^{it}) \rangle \, dm(t)
\]

\[
= \inf_{T(t) \in \mathbb{H}^2} \int_0^{2\pi} \langle w(e^{it})[c - e^{it}f(e^{it})], c - e^{it}f(e^{it}) \rangle \, dm(t)
\]

\[
= \inf_{T(t) \in \mathbb{H}^2} \langle T_0[c(\cdot) - S_0f(\cdot)], c(\cdot) - S_0f(\cdot) \rangle_{\mathbb{H}^2}
\]

\[
= \inf_{T(t) \in \mathbb{H}^2} \langle T(c - Sf), c - Sf \rangle_{\mathbb{H}^2}.
\]

We are able to take the infimum over all of \( H^2 \) since \( w(\cdot) \in L^\infty(\mathbb{B}(\mathbb{C})) \).

If we let \( Y = +T^{1/2} \) and employ the results and notation of the preceding section, then

\[
\langle T[c - Sf], c - Sf \rangle_{\mathbb{H}^2} = \langle Ec - S_0f, Ec - S_0f \rangle_{\mathbb{H}^2}.
\]

But Rng \( E \) is dense in \( \mathbb{H}(Y) \), and \( S_Y \) is an isometry, so that \( S_Y(E\mathbb{H}) \) is dense in Rng \( S_Y \). Thus

\[
d(w(\cdot), c) = \inf_{T(t) \in \mathbb{H}^2} \langle Ec - S_0f, Ec - S_0f \rangle_{\mathbb{H}^2} = \| P_Y(Ec) \|^2_{\mathbb{H}^2},
\]

where \( P_Y \) is the orthogonal projection of \( \mathbb{H}(Y) \) onto \( \mathbb{C}_Y \).

We now give a necessary and sufficient condition for \( \| P_Y(Ec) \|^2_{\mathbb{H}^2} > 0 \) for any nonzero vector \( c \) in \( \mathbb{C} \).

**Theorem 4.1.** Let \( T \) be a nonnegative \( S \)-Toeplitz operator in \( \mathbb{B}(\mathbb{H}) \). Set \( Y = +T^{1/2} \) and \( \mathbb{C} = \text{Ker } S^* \). Suppose \( c \) is any nonzero vector in \( \mathbb{C} \) then \( \| P_Y(Ec) \|_{\mathbb{H}^2} > 0 \), if and only if, \( \mathbb{C} \cap \text{Rng } Y \) is dense in \( \mathbb{C} \).

**Proof.** Suppose \( \mathbb{C} \cap \text{Rng } Y \) is not dense in \( \mathbb{C} \). Therefore, there exists a nonzero vector \( c \) in \( \mathbb{C} \) such that \( c \) is in \( [F(\mathbb{C}_Y^+)]^\perp \). Thus

\[
\langle Ec\mathbb{C}_Y \rangle_Y = \langle c, F^*U(\mathbb{C}_Y^+) \rangle_{\mathbb{H}^2} = \langle c, F\mathbb{C}_Y^+ \rangle_{\mathbb{H}^2} = 0.
\]

Now assume \( \mathbb{C} \cap \text{Rng } Y \) is dense in \( \mathbb{C} \). For any nonzero vector \( c \) in \( \mathbb{C} \) there exists a sequence of vectors \( \{c_n\}_1^\infty \subseteq F(\mathbb{C}_Y^+) \) such that \( c_n \to c \) as \( n \to \infty \). Suppose for every vector \( c_Y \in \mathbb{C}_Y \) we have \( \langle c_Y, Ec \rangle_Y = 0 \). If this be the case, then

\[
0 = \langle UFc_n, Ec \rangle_Y = \langle Fc_n, Fc \rangle_Y = \langle c_n, c \rangle_{\mathbb{H}^2} \quad \text{for every } n.
\]

Contradiction.

**Corollary 4.2 [Szegö].** Let \( w(\cdot) \) be a nonnegative, essentially bounded, scalar function on the unit circle. Denote by \( T_0 \) the associated
nonnegative classical Toeplitz operator on $H^2$ and $Y_0$ the nonnegative square root of $T_0$. If 1 is in Rng $Y_0$, then

$$\inf_{f(t) \in A} \int_0^{2\pi} |1 - e^{it} f(e^{it})|^2 w(e^{it}) dm(t) = \|Y_0^+(1)\|^{-2} = \exp \int_0^{2\pi} \log w(e^{it}) dm(t).$$

**Proof.** Let $H = \Phi^* \oplus H^2$ where $\Phi_0$ is the Fourier representation, Dim $\mathbb{C} = 1$, and let $Y = \Phi^* \oplus \Phi_0 \Phi_0$. By the opening remarks of this section we have

$$\inf_{f(t) \in A} \int_0^{2\pi} |1 - e^{it} f(e^{it})|^2 w(e^{it}) dm(t) = \|P_Y(E_0)\|^2, \quad (c = \Phi^*(1)).$$

In addition,

$$\sup_{k \in \mathbb{C}} \frac{\langle k, E_0 \rangle}{\|k\|} = \frac{\langle UF^+c, E_0 \rangle}{\|F^+c\|} = \|Y^+(c)\|^{-1}.$$ 

Since $\|Y^+(c)\|_{\mathbb{C}} = \|Y_0^+(1)\|$, we have the first equality.

Let $E(\cdot)$ be the resolution of identity of $Y_0$. Because 1 is in Rng $Y_0$, we have $1 \perp \text{Ker } Y_0$, and

$$\lim_{\epsilon \downarrow 0} \langle (T_0 + \epsilon)^{-1} 1, 1 \rangle = \lim_{\epsilon \downarrow 0} \int_0^{2\pi} (\lambda^2 + \epsilon)^{-1} d\langle E_0 1, 1 \rangle = \int_0^{\infty} (\lambda^2)^{-1} d\langle E_0 1, 1 \rangle.$$

But $\langle E_0 1, 1 \rangle = 0$, so that

$$\int_0^{\infty} (\lambda^2)^{-1} d\langle E_0 1, 1 \rangle = \int_0^{\infty} \lambda^2 d\langle E_0 1, 1 \rangle = \|Y_0^+(1)\|^2 = 0,$$

where $c(\lambda) = 1/\lambda$, $\lambda \neq 0$ and $c(0) = 0$.

From Rosenblum [13, p. 592] the following formula can be obtained:

$$\langle (T_0 + \epsilon)^{-1} k(u), k(v) \rangle = (1 - u \epsilon)^{-1} \exp - \int_0^{2\pi} \log(w(e^{it}) + \epsilon) [P(u, t) + P^*(v, t)] dm(t),$$

where $k(u) \in L^2$, $k_n(u) = u_n$, $k_0(0) = 1$, $u$ in the unit disk and $P(u, t) = (4\pi)^{-1}(1 + u e^{it})(1 - u e^{it})^{-1}$. Here we identify $H^2$ with $L^2$ and consider, by an abuse of notation, $T_0$ to be defined on the sequence space.

Taking limits yields
\[
\exp \int_0^{2\pi} \log w(e^{it}) \, dm(t) = ||V_0^+(1)||^{-2}.
\]

5. **Remark.** The techniques of this paper can be extended via closed and densely defined \(S\)-Toeplitz operators \(T\), thereby including more general weight functions.

Certain domain demands must be made, namely \(C \subseteq \text{Dom } T^{1/2}\). \(\mathcal{H}(T^+)\) is not complete [9] if \(\mathcal{H}\) is only closed and densely defined and, hence, an abstract completion is necessary. Further, in this case, the operators \(E\) and \(F\) are closed and densely defined.

Finally, we only used the identification between the abstract shift \(S\) and \(S_0\) on \(H_0^\infty\). By identifying \(S\) with the translation of \(L_0^\infty[0, \infty)\), we encompass, in a prediction theoretic sense, the continuous and discrete stationary processes.

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**References**


University of New Hampshire, Durham, New Hampshire 03824