COEFFICIENTS OF FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

JAMES W. NOONAN

Abstract. For k ≥ 2 denote by \( V_k \) the class of normalized functions analytic in the unit disc which have boundary rotation at most \( k\pi \). For fixed \( n \leq (k+6)/4 \) we determine the maximum of the set of values of \( |a_n| \), where \( a_n \) is the \( n \)th Taylor coefficient of a function in \( V_k \).

For a fixed \( k \geq 2 \) let \( V_k \) denote the class of functions

\[
f(z) = z + a_2z^2 + a_3z^3 + \cdots
\]

which are analytic in \( U = \{ z : |z| < 1 \} \) and have an integral representation of the form

\[
f'(z) = \exp \left\{ \frac{1}{\pi} \int_0^{2\pi} \log(1 - ze^{-it})^{-1} d\mu(t) \right\},
\]

where \( \mu(t) \) is real-valued and of bounded variation on \([0, 2\pi]\) with

\[
\int_0^{2\pi} d\mu(t) = 2\pi, \quad \int_0^{2\pi} |d\mu(t)| \leq k\pi.
\]

V. Paatero [7] showed that \( f(z) \) given by (1) belongs to \( V_k \) if and only if \( f'(z) \neq 0 \) in \( U \) and \( f(z) \) maps \( U \) onto a domain with boundary rotation at most \( k\pi \). (See [4] for a definition of this concept.)

\( V_2 \) is precisely the class of normalized univalent functions that map \( U \) onto a convex domain, and it is known [7] that for \( 2 \leq k \leq 4 \), \( V_k \) consists only of univalent functions. It is also known [1] that \( f(z) \in V_k \) if and only if there exist \( s_1(z) \) and \( s_2(z) \in S^* \), the class of normalized starlike functions, such that

\[
f'(z) = \frac{(s_1(z)/z)^{(k+3)/4}}{(s_2(z)/z)^{(k-3)/4}}.
\]
This follows directly from (2).

In spite of considerable effort, the problem of determining \( M(n, k) = \max_{f \in \mathcal{V}_k} |a_n| \) where \( f(z) \) is given by (1) has been solved for all \( n \) only when \( k = 2 \) and \( k = 4 \). K. Löwner [6] proved that \( M(n, 2) = 1 \), and A. Rényi [8] proved that \( M(n, 4) = n \). O. Lehto [4] proved that \( M(2, k) = k/2 \) and \( M(3, k) = (k^2 + 2)/6 \). It is also known [2] that \( M(4, k) = (k^3 + 8k)/24 \). The extremal function in all of these cases has been

\[
F_k(z) = \frac{1}{k} \left\{ \left( \frac{1 + z}{1 - z} \right)^{k/2} - 1 \right\}.
\]

It is also known ([3], [9]) that \( M(n, k) \leq C(k)n^{k/2-1} \) where \( C(k) \) depends only on \( k \), and [4] that

\[
M(n, k) \sim k^{n-1}/n! \quad (k \to \infty).
\]

Again \( F_k(z) \) is extremal. These facts tend to support the conjecture that \( F_k(z) \) is the solution of the coefficient problem \( M(n, k) \) for all \( n \) and for all \( k \). The purpose of this paper is to show that if \( n \leq \left\lceil (k + 6)/4 \right\rceil \), then \( F_k(z) \) is the solution to the problem \( M(n, k) \).

Before we can prove this, however, we need two lemmas. The proofs of these lemmas are modeled after that of Lemma 4 [5, p. 169], which in turn is due to Professor M. S. Robertson.

**Lemma 1.** Let \( s(z) \in \mathcal{S}^* \) and let \( \beta > 0 \). Let \( g(z) = (s(z)/z)^{\beta} = \sum_{n=0}^{\infty} b_n z^n \). Then for \( 0 \leq n \leq \beta + 1 \) we have

\[
|b_n| \leq \frac{\Gamma(2\beta + 1)}{\Gamma(n + 1) \Gamma(2\beta - n + 1)},
\]

where \( \Gamma \) represents the gamma function.

**Proof.** From the definition of \( g(z) \) we have

\[
\log g(z) = -\beta \log \frac{s(z)}{z}.
\]

By differentiating both sides of this equation, we see that

\[
1 - zg'(z)/\beta g(z) = zs'(z)/s(z).
\]

Since \( s(z) \in \mathcal{S}^* \), we see that

\[
P(z) = 1 - zg'(z)/\beta g(z)
\]

belongs to \( \mathcal{P} \), the class of normalized functions with positive real part. Therefore, \( (P(z))^{-1} = 1 + \sum_{m=1}^{\infty} \mu_m z^m \in \mathcal{P} \), and \( |\mu_m| \leq 2 \) for all \( m \).
Now \((P(z))^{-1}(\beta g(z) - zg'(z)) = \beta g(z)\). Thus by equating coefficients of \(z^n\) we see that

\[
nb_n = \sum_{m=0}^{n-1} (\beta - m) b_m \mu_{n-m}.
\]

A simple induction argument, in which we use the fact that \(|\mu_{n-m}| \leq 2\), now completes the proof. The requirement that \(n \leq \beta + 1\) is used in the induction argument to insure that \(\beta - (n - 1) \geq 0\).

**Lemma 2.** Let \(s(z) \in S^*\) and let \(\alpha > 0\). Let \(g(z) = (s(z)/z)^\alpha = \sum_{n=0}^\infty b_n z^n\). Then

\[
|b_n| \leq \frac{\Gamma(n + 2\alpha)}{\Gamma(n + 1) \Gamma(2\alpha)}.
\]

**Proof.** As in the proof of Lemma 1 we see that

\[
\log g(z) = \alpha \log \frac{s(z)}{z},
\]

which shows that

\[
P(z) = 1 + \frac{s(g'(z))}{\alpha g(z)} = 1 + \sum_{n=1}^\infty \delta_n z^n
\]

belongs to \(S^*\). Thus \(|\delta_n| \leq 2\).

By expanding \(g(z)\) and \(g'(z)\) as power series and then comparing coefficients of \(z^n\), we see that

\[
nb_n = \alpha \sum_{m=0}^{n-1} b_m \delta_{n-m}.
\]

The proof is then completed by induction. Note that this lemma is true for all \(n\).

We are now able to prove the following theorem on the coefficients of a \(V_k\) function.

**Theorem.** Let \(f(z) = z + \sum_{n=2}^\infty a_n z^n \in V_k\), and let \(F_k(z) = z + \sum_{n=2}^\infty A_n(k) z^n\) be given by (4). Then for \(n \leq [(k+6)/4]\) we have \(|a_n| \leq |A_n(k)|\) with equality for any \(n \leq [(k+6)/4]\) if and only if \(f(z) = e^{\theta} F_k(e^{-\theta z})\) for some \(\theta \in [0, 2\pi]\).

**Proof.** By (3) we have that

\[
f'(z) = \frac{(s_1(z)/z)^{(k+2)/4}}{(s_2(z)/z)^{(k-2)/4}}
\]
where $s_1(z), s_2(z) \in S^*$. Let

$$(s_1(z)/z)^{(k+2)/4} = \sum_{j=0}^{\infty} c_j z^j$$

and

$$(s_2(z)/z)^{-(k-2)/4} = \sum_{j=0}^{\infty} b_j z^j.$$  

Then

$$f'(z) = \sum_{m=1}^{\infty} m a_m z^{m-1} = \left( \sum_{j=0}^{\infty} c_j z^j \right) \left( \sum_{h=0}^{\infty} b_h z^h \right).$$

By equating coefficients we see that

$$(n + 1) a_{n+1} = c_0 b_n + c_1 b_{n-1} + \cdots + c_{n-1} b_1 + c_n b_0.$$  

From Lemma 2, with $\alpha = (k+2)/4$, we have that

$$|c_j| \leq \frac{\Gamma(j + k/2 + 1)}{\Gamma(j + 1) \Gamma(k/2 + 1)} \quad (j \geq 0)$$

and from Lemma 1, with $\beta = (k-2)/4$, we have that

$$|b_m| \leq \frac{\Gamma(k/2)}{\Gamma(m + 1) \Gamma(k/2 - m)} \quad (0 \leq m \leq (k + 2)/4).$$

Therefore, for $n \leq [(k+2)/4]$ we have that

$$(n + 1) |a_{n+1}| \leq \sum_{j=0}^{n} \frac{\Gamma(j + k/2 + 1)}{\Gamma(j + 1) \Gamma(k/2 + 1)} \frac{\Gamma(k/2)}{\Gamma(n - j + 1) \Gamma(k/2 - (n - j))}.$$  

(6)

Now let us examine

$$F_k'(z) = \frac{(1 - z)^{-(k+2)/2}}{(1 + z)^{-(k-2)/2}}.$$  

Let $(1 - z)^{-(k+2)/2} = \sum_{n=0}^{\infty} C_n z^n$ and $(1 + z)^{-(k-2)/2} = \sum_{m=0}^{\infty} B_m z^m$. Then

$$C_n = \frac{\Gamma(n + k/2 + 1)}{\Gamma(n + 1) \Gamma(k/2 + 1)} \quad (n \geq 0)$$

and
Then we see that

\[(n + 1)A_{n+1}(k) = C_0B_n + C_1B_{n-1} + \cdots + C_{n-1}B_1 + C_nB_0.\]

Since we require that \(n \leq \lfloor (k+2)/4 \rfloor\), we have also that \(n \leq \lfloor k/2 \rfloor\).

Therefore

\[(n + 1)A_{n+1}(k) = \sum_{j=0}^{n} \frac{\Gamma(j + k/2 + 1)}{\Gamma(j + 1)\Gamma(k/2 + 1)} \cdot \frac{\Gamma(k/2)}{\Gamma(n - j + 1)\Gamma(k/2 - (n - j))} (n \leq \lfloor (k+2)/4 \rfloor).\]

Comparing (6) with (7), we find that

\[|a_{n+1}| \leq |A_{n+1}(k)| (n \leq \lfloor (k+2)/4 \rfloor).\]

This is equivalent to the statement of the theorem.

If equality holds for any \(n\), then an examination of the above proof shows that \(|a_2| = A_2(k) = k/2\). But it is well-known [4] that \(|a_2| = k/2\) if and only if \(f(z) = e^{i\theta}F_k(e^{-i\theta}z)\) for some \(\theta \in [0, 2\pi]\). This completes the proof of the theorem.

This theorem allows us to improve Lehto's result (5) in the following sense.

**Corollary.** Let \(F_k(z) = z + \sum_{n=2}^\infty A_n(k)z^n\) be given by (4). Let \(n\) be fixed but arbitrary, \(n \geq 2\). Then for \(k \geq 4n - 6\) we have \(M(n, k) = A_n(k)\). In particular, for any fixed \(n\), \(M(n, k) \sim k^{n-1}/n! (k \to \infty)\).

**Proof.** Since \(k \geq 4n - 6\), we have \(n \leq \lfloor (k+6)/4 \rfloor\). Then, by the theorem, we see that \(M(n, k) = A_n(k)\). Direct computation then shows that \(M(n, k) \sim k^{n-1}/n! (k \to \infty)\).

**Bibliography**


University of Maryland, College Park, Maryland 20742

E. O. Hulburt Center for Space Research, Naval Research Laboratory, Washington, D. C.