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STRICTLY CYCLIC WEIGHTED SHIFTS[1]

ALAN LAMBERT

ABSTRACT. E. Nordgren recently showed that certain weighted shift operators on Hilbert space have the property that they commute with no unbounded closed densely defined linear transformations. In the present paper, it is shown that this property for a weighted shift operator is equivalent to the existence of a strictly cyclic vector for the weakly closed algebra generated by the weighted shift. After establishing some tests for the existence of such a strictly cyclic vector, several further examples are given of weighted shift operators satisfying Nordgren's commutativity property.

1. **Introduction.** Let $\mathcal{H}$ be a separable complex Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. For each bounded sequence $\alpha=\{\alpha_n\}_{n=1}^{\infty}$ of nonzero scalars, the bounded operator S_α on $\mathcal{H}$ defined by $S_\alpha e_n =\alpha_{n+1}e_{n+1}$ is called the (forward) weighted shift with weight sequence $\{\alpha_n\}$. E. Nordgren proved in [2] that if $\{|\alpha_n|\}$ is monotonically decreasing and $\sum|\alpha_n|^2<\infty$, then

(i) every closed densely defined linear transformation commuting with S_α is bounded; and

(ii) every bounded operator commuting with S_α is in the uniformly closed algebra generated by S_α.

The purpose of this paper is to give necessary and sufficient conditions for a shift S_α to have property (i) and to show every shift with property (i) also has property (ii). We then give some further examples of shifts with property (i).

2. **Preliminaries.** Throughout this paper S_α will denote a weighted shift with weight sequence $\{\alpha_n\}$. Set $\beta_0=1$ and $\beta_n=\prod_{k=1}^{n}\alpha_k$ for $n\geqq 1$. It is shown in [1] that a bounded operator A commutes with S_α if and only if A has matrix $(\lambda_{ij})_{i,j=0}$ relative to $\{e_n\}$, where

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$$\lambda_{ij} = 0 \qquad \text{for } i < j,$$
$$(1) \qquad = \beta_i/(\beta_j\beta_{i-j})\lambda_{i-j\,0} \quad \text{for } i \geqq j.$$

Let $\mathfrak{A}$ be the (nonselfadjoint) weakly closed algebra generated by S_α. A. L. Shields and L. J. Wallen [3] have shown that $\mathfrak{A}$ is the algebra of all bounded operators commuting with S_α. It follows that $\mathfrak{A}$ is a maximal abelian subalgebra of $\mathfrak{L}(\mathfrak{H})$. Since $S_\alpha^n e_0 = \beta_n e_n$, e_0 is cyclic for $\mathfrak{A}$, i.e., $\mathfrak{A}e_0 = \{Ae_0 : A \text{ in } \mathfrak{A}\}$ is dense in $\mathfrak{H}$. Thus for each x in $\mathfrak{A}e_0$ there is a unique operator A_x in $\mathfrak{A}$ such that $A_x e_0 = x$. Let ρ be the mapping $x \to A_x$. Then ρ is a bijective linear transformation from $\mathfrak{A}e_0$ onto $\mathfrak{A}$.

For each $x = \sum_{n=0}^\infty \lambda_n e_n$ in $\mathfrak{H}$, let U_x be the linear transformation defined by (1) with $\lambda_{k0} = \lambda_k$. Since U_x is matricially defined, U_x is a closed linear transformation. Moreover, if x is in $\mathfrak{A}e_0$, then $U_x = A_x$, and for any x, if y is in the domain of U_x, then x is in the domain of U_y. It follows that for any x in $\mathfrak{H}$, $\mathfrak{A}e_0$ is contained in the domain of U_x and for y in $\mathfrak{A}e_0$, $U_x y = A_y x$.

For any linear transformation T, let $D(T)$ denote the domain of T. If A is a bounded operator then by "T commutes with A" we mean $A(D(T))$ is contained in $D(T)$ and $AT = TA$ on $D(T)$.

LEMMA 2.1. *For each x in $\mathfrak{H}$, U_x commutes with S_α.*

PROOF. Let $x = \sum_{n=0}^\infty \lambda_n e_n$ be in $\mathfrak{H}$ and let $y = \sum_{n=0}^\infty \gamma_n e_n$ be in $D(U_x)$. Then $S_\alpha y = \sum_{n=0}^\infty \delta_n e_n$, where $\delta_0 = 0$ and $\delta_n = \alpha_n \gamma_{n-1}$ for $n \geqq 1$. Also,

$$\begin{aligned}
SU_x y &= \sum_{n=0}^\infty \left(\sum_{m=0}^n \frac{\beta_n}{\beta_m\beta_{n-m}} \alpha_{n+1}\lambda_{n-m}\gamma_m \right) e_{n+1} \\
&= \sum_{n=0}^\infty \left(\sum_{m=0}^n \frac{\beta_{n+1}}{\beta_m\beta_{n-m}} \lambda_{n-m}\gamma_m \right) e_{n+1} \\
&= \sum_{n=1}^\infty \left(\sum_{m=1}^n \frac{\beta_n}{\beta_m\beta_{n-m}} \alpha_m\lambda_{n-m}\gamma_{m-1} \right) e_n \\
&= \sum_{n=0}^\infty \left(\sum_{m=0}^n \frac{\beta_n}{\beta_m\beta_{n-m}} \lambda_{n-m}\delta_m \right) e_n,
\end{aligned}$$

showing $S_\alpha y$ is in $D(U_x)$ and $U_x S_\alpha y = S_\alpha U_x y$.

We will say S_α is *strictly cyclic* if $\mathfrak{A}e_0 = \mathfrak{H}$. Clearly, S_α is strictly cyclic if and only if U_x is bounded for every x in $\mathfrak{H}$.

LEMMA 2.2. *S_α is strictly cyclic if and only if ρ is continuous with respect to the strong topology on $\mathfrak{A}e_0$ and the uniform topology on $\mathfrak{A}$.*

PROOF. We show first that for any shift S_α, ρ^{-1} is continuous. Let $\{A_{x_n}\}$ be a sequence in $\mathcal{A}$ converging in norm to A_x. Then $x_n = A_{x_n}e_0$ converges strongly to $A_x e_0 = x$. It follows that ρ is a closed linear transformation. Thus if $\mathcal{A}e_0 = \mathcal{H}$, then ρ is continuous by the closed graph theorem. Conversely, suppose ρ is continuous. Then there is a constant M such that for every x in $\mathcal{A}e_0$, $\|A_x\| \leq M\|x\|$. Now, for each x in $\mathcal{H}$, there is a sequence $\{x_n\}$ in $\mathcal{A}e_0$ converging strongly to x. But then $\{A_{x_n}\}$ is a Cauchy sequence and so converges to an operator A in $\mathcal{A}$. Since $Ae_0 = \text{limit}_{n\to\infty} x_n = x$, S_α is strictly cyclic.

COROLLARY 2.3. *If S_α is strictly cyclic, then the strong and uniform operator topologies on $\mathcal{A}$ are identical.*

PROOF. If $\{A_{x_\lambda}\}$ is a net in $\mathcal{A}$ converging strongly to A_x, then $\{x_\lambda\}$ converges strongly to x. By Lemma 2.2, $\{A_{x_\lambda}\}$ converges uniformly to A_x.

3. We will show in this section that for a shift S_α property (i) is equivalent to strict cyclicity. We first prove a series of lemmas.

LEMMA 3.1. *If T is a closed densely defined linear transformation commuting with S_α, then T commutes with every operator in $\mathcal{A}$.*

PROOF. Let A be in $\mathcal{A}$. By [3] there is a sequence $\{P_n\}$ of polynomials in S_α converging to A in the weak operator topology. Let x be in $D(T)$. Then for every y in $D(T^*)$,

$$(ATx, y) = \lim_{n\to\infty} (TP_n x, y) = \lim_{n\to\infty} (P_n x, T^*y) = (Ax, T^*y),$$

showing Ax is in $D(T^{**}) = D(T)$ and $TAx = ATx$.

LEMMA 3.2. *If S_α is strictly cyclic, then every closed densely defined linear transformation commuting with S_α is bounded.*

PROOF. Let T be a closed densely defined linear transformation commuting with S_α. By Lemma 3.1, $\mathcal{A}D(T)$ is contained in $D(T)$. We will show that $D(T) = \mathcal{H}$ and so by the closed graph theorem T is bounded. Since $D(T)$ is dense and ρ is continuous, there exists an x in $D(T)$ such that $\|A_x - I\| < 1$ (using the fact that $A_{e_0} = I$). In particular A_x is invertible. Since A_x^{-1} commutes with S_α, A_x^{-1} is in $\mathcal{A}$. It follows that A_x^{-1} is in $D(T)$. But $A_x^{-1}x = A_x^{-1}A_x e_0 = e_0$. Thus for every y in $\mathcal{H}$, $y = A_y e_0$ is in $D(T)$.

Define $\mathfrak{F}$ to be the set of all linear functionals on $\mathcal{A}$ of the form $A_x \to (x, y)$, as y ranges over $\mathcal{H}$.

LEMMA 3.3. *S_α is strictly cyclic if and only if the dual space of $\mathfrak{A}$ is exactly $\mathfrak{F}$.*

PROOF. Suppose first that S_α is strictly cyclic. Let f be a continuous linear functional on $\mathfrak{A}$. Then $f \circ \rho$ is a continuous linear functional on $\mathfrak{H}$. Thus there is a vector y in $\mathfrak{H}$ such that for every x in $\mathfrak{H}$, $f(A_x) = f \circ \rho(x) = (x,y)$. Conversely, suppose the dual space of $\mathfrak{A}$ is $\mathfrak{F}$. Then for each pair x and y of vectors in $\mathfrak{H}$ there is a vector $K_x y$ in $\mathfrak{H}$ such that $(Ax, y) = (Ae_0, K_x y)$ for every A in $\mathfrak{A}$. Since $\mathfrak{A}e_0$ is strongly dense in $\mathfrak{H}$, $K_x y$ is uniquely defined. It is easy to see that the map $y \rightarrow K_x y$ is an everywhere defined linear transformation on $\mathfrak{H}$ for each x in $\mathfrak{H}$. Fix x in $\mathfrak{H}$. Then for z in $\mathfrak{A}e_0$ and y in $\mathfrak{H}$,

$$(z, K_x y) = (U_x z, y).$$

This implies that $\mathfrak{A}e_0$ is contained in $D(K_x^*)$ and $U_x = K_x^*$ on $\mathfrak{A}e_0$. But since K_x is everywhere defined and K_x^* is densely defined, K_x is bounded. It follows that U_x is a closed linear transformation agreeing with a bounded operator on a dense set and so U_x is bounded. Since x was arbitrary, S_α is strictly cyclic.

REMARK. We proved (Lemma 3.2) that if S_α is strictly cyclic, then every closed densely defined linear transformation commuting with S_α is bounded. The converse is of course also true, since U_x is closed for every x in $\mathfrak{H}$. Thus we have proved the following theorem.

THEOREM 3.4. *The following are equivalent*:

(a) *S_α is strictly cyclic.*

(b) *ρ is continuous.*

(c) *Every closed densely defined linear transformation commuting with S_α is bounded.*

(d) *The dual space of $\mathfrak{A}$ is exactly $\mathfrak{F}$.*

4. **Examples of strictly cyclic shifts.** We now state and prove a theorem that yields several examples of strictly cyclic shifts.

THEOREM 4.1. *Let S_α be a weighted shift with β_n defined as in* §2. *If there exist square summable sequences of positive numbers $\{\mu_n\}_{n=0}^\infty$ and $\{\nu_n\}_{n=0}^\infty$ such that for all nonnegative integers n and m,*

$$\left|\frac{\beta_{n+m}}{\beta_n\beta_m}\right| \leqq \mu_n + \nu_m,$$

then S_α is strictly cyclic.

PROOF. Since any two separable complex Hilbert spaces are unitarily equivalent, there is no loss in generality in assuming $\mathfrak{H}$ is H^2

of the circle with orthonormal basis $\{e_n\}_{n=0}^{\infty}$, $e_n(z)=z^n$. Also, since two weighted shifts S_α and S_η are unitarily equivalent if and only if $|\alpha_n|=|\eta_n|$ for all n (see [1]), we may assume $\alpha_n>0$ for all n. Finally, to show S_α is strictly cyclic it suffices to show g is in $D(U_f)$ for every pair of vectors f and g in H^2 where both f and g have nonnegative fourier coefficients. Let $f=\sum_{n=0}^{\infty}\lambda_n e_n$ and $g=\sum_{n=0}^{\infty}\gamma_n e_n$ be in H^2, $\lambda_n, \gamma_n \geqq 0$ for all n. The power series

$$\sum_{n=0}^{\infty}\lambda_n\mu_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty}\gamma_n\nu_n z^n$$

define functions analytic in the open unit disk and continuous on the closed unit disk. In particular,

$$f_1=\sum_{n=0}^{\infty}\lambda_n\mu_n e_n \quad \text{and} \quad g_1=\sum_{n=\theta}^{\infty}\gamma_n\nu_n e_n$$

are in H^∞. It follows that g_1f and f_1g are in H^2 with

$$g_1f=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\nu_{n-m}\gamma_{n-m}\lambda_m\right)e_n$$

and

$$f_1g=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\mu_m\lambda_m\gamma_{n-m}\right)e_n.$$

We then have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\beta_n}{\beta_m\beta_{n-m}}\lambda_m\gamma_{n-m}\right)^2\\
&\qquad\leqq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(\mu_m\lambda_m\gamma_{n-m}+\nu_{n-m}\gamma_{n-m}\lambda_m)\right)^2\\
&\qquad\leqq 2\left(\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\mu_m\lambda_m\gamma_{n-m}\right)^2+\left(\sum_{m=0}^{n}\nu_{n-m}\gamma_{n-m}\lambda_m\right)^2\right)\\
&\qquad< \infty.
\end{aligned}$$

COROLLARY 4.2. *If $\{\gamma_n\}$ is a monotonically decreasing sequence of positive numbers, then the shift with weight sequence $\alpha=\{\gamma_n(n+1)/n\}$ is strictly cyclic.*

PROOF. Let $\mu_n=\prod_{k=1}^{n}\gamma_k$ for $n\geqq 1$ and $\mu_0=1$. Then $\mu_{n+m}/\mu_n\mu_m\leqq 1$ so that $\beta_{n+m}/\beta_n\beta_m\leqq 1/(n+1)+1/(m+1)$.

COROLLARY 4.3. *If for all n, $\alpha_n \geqq (1+1/n^2)\alpha_{n+1}$, then S_α is strictly cyclic.*

PROOF. A short computation shows $\alpha_n n/(n+1)$ is monotonically decreasing. By Corollary 4.2, S_α is strictly cyclic.

REMARKS. The strict cyclicity of a shift with monotone decreasing, square summable weights can be deduced directly from Theorem 4.1. For if S_α is such a shift, then it is easy to see that

$$\left| \beta_{n+m}/\beta_n\beta_m \right| \leqq \left| \alpha_n/\alpha_1 \right| .$$

Also, using Corollary 4.2, it is easily shown that the shifts with weight sequences $\{(n+1)/n\}$ and $\{1/\log(n+1)\}$ are strictly cyclic. The former is an example of a nonquasinilpotent strictly cyclic shift, and the latter has the property that its weights converge to zero monotonically but lie in no l^p class.

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UNIVERSITY OF MICHIGAN, ANN ARBOR, MICHIGAN 48104