**IRREDUCIBLE LIE ALGEBRAS OF INFINITE TYPE**

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**Abstract.** Let $V$ be a finite dimensional vector space over an algebraically closed field of characteristic $\not\equiv 2, 3, 5$. It is shown that if $L \subseteq \mathfrak{gl}(V)$ is an irreducible Lie algebra of infinite type then either $L = \mathfrak{gl}(V)$, $L = \mathfrak{sl}(V)$, $\dim V = 2r \geq 4$ and $L = \mathfrak{sp}(V)$, $\dim V = 2r \geq 4$ and $L = \mathfrak{csp}(V)$, or there exists $A \in L$ such that $\text{ad } A = 0 = (\text{ad } A)^2$.

As a corollary we obtain E. Cartan's classification of the irreducible Lie algebras of infinite type over $\mathbb{C}$.

Let $L$ be a Lie algebra of linear transformations of a vector space $V$. For each nonnegative integer $n$ the $n$th Cartan prolongation, $L_n$, is defined inductively by $L_0 = L$ and

$$L_n = \left\{ \phi \in \text{Hom}(V, L_{n-1}) \mid \gamma(x\phi) = x(\gamma \phi) \text{ for all } x, y \in V \right\}$$

for $n \geq 1$. If $L_n \neq 0$ for all $n \geq 0$ then $L$ is said to be of infinite type. The main result of this paper is:

**Theorem 1.** Let $V$ be a finite dimensional vector space over an algebraically closed field of characteristic $\not\equiv 2, 3, 5$. If $L$ is an irreducible Lie algebra of infinite type then either $L = \mathfrak{gl}(V)$, $L = \mathfrak{sl}(V)$, $\dim V = 2r \geq 4$ and $L = \mathfrak{sp}(V)$, $\dim V = 2r \geq 4$ and $L = \mathfrak{csp}(V)$, or there exists an $A \in L$ such that $\text{ad } A = 0 = (\text{ad } A)^2$.

Now it is easily seen (as in [7]) that if $(\text{ad } A)^2 = 0$ then $A$ belongs to the radical of the Killing form. If $\Phi = \mathbb{C}$ and $L$ is reductive this implies that $\text{ad } A = 0$. Furthermore, it is known (Theorem 1 of [4]) that an irreducible Lie algebra over $\mathbb{C}$ is of infinite type if and only if $L_2 \neq 0$. Thus Theorem 1 implies the following theorem of E. Cartan [1]:

**Theorem 2.** Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and let $L$ be an irreducible Lie algebra of linear transformations of $V$ such that $L_2 \neq 0$. Then either $L = \mathfrak{gl}(V)$, $L = \mathfrak{sl}(V)$, $\dim V = 2r \geq 4$ and $L = \mathfrak{sp}(V)$, or $\dim V = 2r \geq 4$ and $L = \mathfrak{csp}(V)$.

Theorem 2, which is important in the study of primitive pseudo-groups and infinite Lie algebras of Cartan type, has been proved by several authors ([1], [2], [3], [5], [9]). These proofs have involved considerable use of the classification and representation theory of
semisimple Lie algebras over $\mathbb{C}$ and hence cannot be generalized to fields of prime characteristic. In the proof presented here we use more elementary techniques which are valid over algebraically closed fields of characteristic $\neq 2, 3, 5$.

We will consider the following three conditions on a Lie algebra $L$ of linear transformations of a vector space $V$:

**Condition A.** There exists $A \in L$ with rank $A = 1$ and $A^2 \neq 0$.

**Condition B.** Either $\dim V = 2$ or there exist $A, B \in L$ with rank $A = \text{rank } B = 1$, $\ker A = \ker B$, and $VA \neq VB$.

**Condition C.** There exists $A \in L$ with $\text{ad } A = (\text{ad } A)^2$.

By Condition $\sim X$ we will mean the negation of Condition $X$.

Theorem 1 is clearly a consequence of the following two lemmas.

**Lemma 1.** If $\Phi$ is an algebraically closed field of arbitrary characteristic, $V$ is a finite dimensional vector space over $\Phi$, and $L \subseteq \text{gl}(V)$ is a Lie algebra of infinite type then $L$ contains a rank one transformation.

**Lemma 2.** Let $V$ be a finite dimensional vector space over a field $\Phi$ of characteristic $\neq 2, 3, 5$. Let $L$ be an irreducible Lie algebra of linear transformations of $V$. Assume that $L$ contains a rank one transformation. Then:

(i) If Condition A holds $L = \text{gl}(V)$.

(ii) If Conditions $\sim A$, $B$, and $\sim C$ hold then $L = \text{sl}(V)$.

(iii) If Conditions $\sim A$, $\sim B$, and $\sim C$ hold then $\dim V = 2r \geq 4$ and $L = \text{sp}(V)$ or $L = \text{csp}(V)$.

It is shown in [2] that if $\Phi = \mathbb{C}$ Lemma 1 is a consequence of Hilbert's Nullstellensatz. The proof given there is in fact independent of the assumption $\Phi = \mathbb{C}$ and could be used to prove our Lemma 1. We will give a somewhat more elementary proof here. Part (i) of Lemma 2 is proved in [6] for the case $\Phi = \mathbb{C}$.

In the proof of Lemma 1 we will need:

**Lemma 3.** Let $\Phi$ be an algebraically closed field of arbitrary characteristic. Let $V$ and $W$ be finite dimensional vector spaces over $\Phi$ with $2 \leq \dim V \leq \dim W$. Let $T$ be a subspace of $\text{Hom}(V, W)$ such that $\dim T \geq \dim W$. Then there exist $\phi \in T$ and $v \in V$ such that $\phi \neq 0$, $v \neq 0$, and $\phi v = 0$.

**Proof.** Suppose $T \subseteq \text{Hom}(V, W)$ is a counterexample to the lemma such that $\dim W$ is minimal among all counterexamples. As $T$ is a counterexample we have, for any nonzero $v \in V$, $\dim W \geq \dim \phi v = \dim T \geq \dim W$. Thus for any $w \in W$ there is a unique $\phi \in T$ such that $\phi v = w$. Then if $v_1, v_2$ are linearly independent elements of $V$ we can find a basis $\{\psi_1, \ldots, \psi_n\}$ of $T$ such that
For if linearly independent elements \( \psi_1, \cdots, \psi_j \) satisfying (1) have been found for some \( 1 \leq j \leq n-1 \) then the minimality of \( \dim W \) implies that \( v_0 \psi_1, \psi_2, \cdots, v_0 \psi_j \). Thus there exists a unique \( \psi_{j+1} \in T \) such that \( \psi_1, \cdots, \psi_{j+1} \) are linearly independent and satisfy (1). Proceeding by induction on \( j \) gives the result. Now there exists \( \lambda \in \Phi \) such that \( \lambda^n - \sum_{i=0}^{n-1} \lambda a_{i+1} = 0 \). Setting \( b_k = \lambda^{n-k} - \sum_{i=0}^{n-k-1} \lambda a_{k+i} \) for \( 1 \leq k \leq n-1 \) we see that \( \psi = \psi_n + \sum_{i=1}^{n-1} \psi b_i \in T \) and \( (v_2 - \lambda n) \psi = 0 \). This contradicts the choice of \( T \) and proves the lemma.

Proof of Lemma 1. For \( \phi \in L_i \) and \( j \leq i \) define

\[
\text{im}_j(\phi) = \langle v_j(\cdots (v_1(\phi) \cdots) | v_1, \cdots, v_j \in V) \rangle.
\]

Now if \( x \in \ker \phi \) and \( y \in V \) then \( 0 = y(x \phi) = x(y \phi) \) so we have \( \ker \phi \subseteq \ker(y \phi) \). Thus if \( \psi \in \text{im}_j(\phi) \) we have \( \ker \psi \supseteq \ker \phi \). Furthermore if \( d_i = \min \{ \text{rank } \phi \mid 0 \neq \phi \in L_i \} \) we have \( d_0 \leq d_1 \leq \cdots \leq \dim V \). Hence there is some integer \( N \) such that \( d_N = d_{N+1} \) for all \( i \geq N \). Now if \( i \geq j \geq 0 \) and \( \phi \in L_{N+i} \) satisfies \( \text{rank } \phi = d_N \) then for \( 0 \neq \psi \in \text{im}_j(\phi) \) we have \( \ker \psi \supseteq \ker \phi \) and \( \text{rank } \psi \geq d_N = \text{rank } \phi \). Thus \( \ker \psi = \ker \phi \).

Thus \( \text{im}_j(\phi) \subseteq \text{Hom}(V/\ker \phi, \text{im}_{j+1}(\phi)) \) and \( 0 \neq \psi \in \text{im}_j(\phi) \) implies \( \text{rank } \psi = d_N = \dim(V/\ker \phi) \). Thus if \( d_N \geq 2 \), Lemma 3 shows that \( \dim \text{im}_{j+1}(\phi) \leq \dim \text{im}_{j+1}(\phi) - 1 \). Hence \( \dim L_{N-1} \geq \dim \text{im}_{j+1}(\phi) \geq \dim \text{im}_j(\phi) + i \geq i \) for all \( i \geq 0 \). Since \( L_{N-1} \) is finite dimensional this is impossible. Hence \( 1 = d_N = d_0 \), proving the lemma.

Proof of Lemma 2. Let \( n = \dim V \). If \( \{x_1, \cdots, x_n\} \) and \( \{y_1, \cdots, y_n\} \) are bases for \( V \) we define elements \( E_{ij} \in \text{gl}(V) \) for \( 1 \leq i, j \leq n \) by \( x_k E_{ij} = \delta_{ik} x_j \) and \( y_k F_{ij} = \delta_{kj} y_i \). If \( 2k \leq n \) we define \( \sp(x_1, \cdots, x_{2k}) \) to be the Lie algebra of all \( A \in \text{gl}(V) \) such that \( VA \subseteq \langle x_1, \cdots, x_{2k} \rangle, x_i A = 0 \) for all \( r > 2k \), and \( A \) is skew with respect to the skew-symmetric bilinear form defined by \( (x_{i+1}, x_j) = \delta_{j,i+1}, (x_{2i+1}, x_j) = -\delta_{j,i+1} \), and \( (x_i, x_j) = 0 \) for all \( 0 \leq i \leq k - 1, 1 \leq j \leq n \), \( 2k < r \leq n \).

We will presently verify the following statements about an irreducible Lie algebra \( L \) of linear transformations of \( V \):

(a) If \( 1 \leq k < n \) and \( E_{ii} \in L \) for all \( 1 \leq i \leq k \) then there is a basis \( \{y_1, \cdots, y_n\} \) of \( V \) such that \( F_{ii} \in L \) for all \( 1 \leq i \leq k + 1 \).

(b) If \( 1 \leq k < n \), \( E_{ii} \in L \) for all \( 1 \leq i \leq n \), and \( E_{ii} \in L \) for all \( 1 \leq i \leq k \) then there is a basis \( \{y_1, \cdots, y_n\} \) of \( V \) such that \( F_{ii} \in L \) for all \( 1 \leq i \leq n \) and \( F_{ii} \in L \) for all \( 1 \leq i \leq k + 1 \).
(c) If \(2 \leq k \leq m \leq n\), \(E_{4i} \in L\) for all \(2 \leq i \leq m\), \(E_{4i} \in L\) for all \(2 \leq i < k\), and \(L\) satisfies Condition \(\sim C\) then there is a basis \(\{y_1, \ldots, y_n\}\) of \(V\) such that \(F_{4i} \in L\) for all \(2 \leq i \leq m\) and \(F_{4i} \in L\) for all \(2 \leq i \leq k\).

(d) If \(3 \leq k < n\) and \(E_{4i}, E_{4j} \in L\) for all \(2 \leq i \leq k\) then there is a basis \(\{y_1, \ldots, y_n\}\) of \(V\) such that \(F_{4i} \in L\) for all \(2 \leq i \leq k+1\) and \(F_{4i} \in L\) for all \(2 \leq i \leq k\).

(e) If \(\text{sp}(x_1, \ldots, x_{2k}) \subseteq L\) where \(\dim V > 2k \geq 2\) and \(L\) satisfies Conditions \(\sim B\) and \(\sim C\) then \(\dim V \geq 2k+2\) and there is a basis \(\{y_1, \ldots, y_n\}\) of \(V\) such that \(\text{sp}(y_1, \ldots, y_{2k+2}) \subseteq L\).

(f) If \(\text{sp}(V) \subseteq L\) and \(L\) satisfies Condition \(\sim B\) then \(L = \text{sp}(V)\) or \(L = \text{csp}(V)\).

Lemma 2 follows immediately from statements (a)-(f). For if Condition \(A\) holds we may choose a basis \(\{x_1, \ldots, x_n\}\) for \(V\) such that \(E_{4i} \in L\). Then by (a), (b), and induction on \(k\) we see that \(L = \text{gl}(V)\), proving (i). If Conditions \(\sim A, \sim B,\) and \(\sim C\) hold we may choose a basis \(\{x_1, \ldots, x_n\}\) of \(V\) such that \(E_{4i} \in L\) and if \(n \geq 3\) we may also arrange that \(E_{4i} \in L\). Then using (c), (d), and induction on \(k\) we see that \(L = \text{sl}(V)\), proving (ii). Finally if Conditions \(\sim A, \sim B,\) and \(\sim C\) hold we may choose a basis \(\{x_1, \ldots, x_n\}\) of \(V\) such that \(E_{4i} \in L\). By (c) we may assume that \(\text{sp}(x_1, x_2) \subseteq L\). Then by (e) and induction on \(k\) we have \(\dim V = 2k \geq 4\) and \(\text{sp}(V) \subseteq L\). Then (f) proves (iii).

We now verify (a)-(f). Throughout we will let \(A = \sum E_{ij}a_{ij}\) where the \(a_{ij} \in \Phi\).

(a): As \(\langle x_1, \ldots, x_k \rangle\) is not an invariant subspace there exists \(A \in L\) such that \(a_{ij} \neq 0\) for some \(1 \leq i \leq k < j \leq n\). We may assume that \(i = 1\) (replacing \(A\) by \(A (\text{ad} E_{1})\) if \(i \neq 1\)) and that \(a_{ij} = 0\) whenever \(r > 1\) or \(j = 1\) (replacing \(A\) by \((A (\text{ad} E_{1}) - A) (\text{ad} E_{1})/2\)). Letting \(\{y_1, \ldots, y_n\}\) be any basis for \(V\) satisfying \(y_i = x_i\) for \(1 \leq i \leq k\), \(y_{k+1} = x_{1} A\), and \(y_j \in (x_{k+1}, \ldots, x_n)\) for \(j > k + 1\) gives the result.

The proof of (b) is similar to that of (a).

(c): First assume that \(\text{ad} E_{1k} = 0\). Then for every \(A\) we have \(0 = A (\text{ad} E_{1k}) = \sum_i (E_{ik}a_{ii} - E_{1i}a_{ki})\). Hence \(a_{ki} = 0\) for all \(i \neq k\) so \(\langle x_k \rangle\) is an invariant subspace, contradicting the irreducibility of \(L\). Hence \(\text{ad} E_{1k} \neq 0\) so by Condition \(\sim C\) we have \((\text{ad} E_{1k})^2 \neq 0\). Since \(A (\text{ad} E_{1k})^2 = E_{1k} (-2a_{k1})\) we have \(E_{1k} \in L\) (ad \(E_{1k}\))^2. Since \(E_{1k}^2 = 0\) we have \((\text{ad} E_{1k})^2 = 0\). Thus Lemma V.8.2 of [8] shows that there exists \(A \in L\) such that \(A (\text{ad} E_{1k}) = 2E_{1k}\) and \(A (\text{ad} E_{1k}) (\text{ad} A) = -2A\). Now

\[
A (\text{ad} E_{1k}) (\text{ad} A) = \sum_{i,j} E_{ij} \left(2a_{k1}a_{ii} - \delta_{i1} \sum_r a_{kr}a_{rij} - \delta_{jk} \sum_r a_{ir}a_{r1}\right).
\]
Thus we have \( a_{kl} = -1 \) and

\[
(2) \quad a_{ij} = -a_{il}a_{kj} + \left( \delta_{ij} \sum_r a_{kr}a_{rj} + \delta_{jk} \sum_r a_{ir}a_{rl} \right) / 2.
\]

Setting \( i = j = 1 \) gives \( 0 = \sum_r a_{kr}a_{r1} \). From this, using (2) to substitute for \( a_{rj} \) in \( \sum_r a_{kr}a_{rj} \), we conclude that \( \sum_r a_{kr}a_{rj} = 0 \) for all \( 1 \leq j \leq n \) and similarly that \( \sum_r a_{ir}a_{rl} = 0 \) for all \( 1 \leq i \leq n \). Thus \( a_{kl} = -a_{il}a_{kj} \) and \( \sum_r a_{kr}a_{rj} = 0 \) for \( 1 \leq i, j \leq n \). Now set \( y_1 = x_kA \), \( y_k = x_k \), and \( y_j = x_j + a_{ij}x_k \) for \( j \neq 1, k \). Then \( F_{kl} = A \in L \), \( F_{ik} = -E_{kl} \in L \), and \( F_{ll} = -E_{kl} \in L \) for \( 2 \leq i \leq m \), \( i \neq k \). Finally for \( 2 \leq i < k \) we have \( F_{ii} = -E_{ii}(ad F_{ii})(ad F_{ii}) - (F_{ii} - F_{kk})a_{ki} \in L \) as required.

(d): As in (a) we may find a basis \( \{ y_1, \ldots, y_n \} \) of \( V \) such that \( F_{ii}, F_{ii} \in L \) for \( 1 \leq i \leq k \) and \( A = \sum F_{ij}b_{ij} \in L \) where \( b_{ij} = \delta_{j,k+1} \). Then

\[
F_{1,k+1} = A(ad F_{11})(ad F_{12})(ad F_{13})(ad F_{14}) + (F_{11} - F_{kk})b_{kk} \in L
\]
as required.

(e): It is well known (for example [8, p. 67]) where we take \( x_{2i+1} = v_{i+1} \) and \( x_{2i+2} = v_{r+i+1} \) for \( 0 \leq i \leq r - 1 \) that \( \text{sp}(x_1, \ldots, x_r) \) is spanned by the following elements and their transposes: \( E_{2i+1,2i+1}, E_{2i+2,2i+2}, E_{2i+1,2i+2} - E_{2i+2,2i+1} \), and \( E_{2i+1,2i+2} + E_{2i+2,2i+1} \) for \( 0 \leq i \neq j \leq r - 1 \). Thus it is easily checked that \( \text{sp}(x_1, \ldots, x_{2k+2}) \) is generated by \( \text{sp}(x_1, \ldots, x_{2k+2}), E_{1,2k+1} - E_{2k+2,2k+1}, \) and \( E_{1,2k+1} - E_{2k+2,2k+2} \).

We first show that for some basis \( \{ y_1, \ldots, y_n \} \) of \( V \) we have \( \text{sp}(y_1, \ldots, y_n) \subseteq L \) and \( A = F_{1,2k+1} + \sum_{i>2k} F_{ij}b_{ij} \in L \) where the \( b_{ij} \) is not an invariant subspace there exists \( A \in L \) such that \( a_{ij} \neq 0 \) for some \( 1 \leq i \leq 2k < j \leq n \). We may assume that \( i = 1 \) (replacing \( A \) by \( A(ad E_{11}) \) if \( i = 2 \), by \( A(ad(E_{1,2r+1} - E_{2r+2,2r+1})) \) if \( i = 2r+1 \), and by \( A(ad(E_{1,2r+2} + E_{2r+1,2r+2})) \) if \( i = 2r+2 \), that \( x_1 = 0 \) unless \( r = 1 \), \( s > 2 \) or \( r > 2 \), \( s = 2 \) (replacing \( A \) by \( A(ad E_{11})(ad E_{11}) - (E_{11} - E_{22})(a_{11} - a_{22}) - E_{11}(2a_{11}) \)), and that \( a_{1s} = a_{s1} = 0 \) for \( s \leq k \) (replacing \( A \) by

\[
A - \sum_{i=1}^{k-1} A((ad E_{2i+1,2i+1}))(ad E_{2i+1,2i+1} + (ad E_{2i+1,2i+1}))(ad E_{2i+1,2i+1})).
\]

Then letting \( \{ y_1, \ldots, y_n \} \) be any basis for \( V \) satisfying \( x_i = y_i \) for \( 1 \leq i \leq 2k \), \( y_{2k+1} = x_1A \), and \( y_j \in (x_{2k+1}, \ldots, x_n) \) for \( j > 2k + 1 \) gives the result.

Now assume that \( \text{sp}(x_1, \ldots, x_{2k}) \subseteq L \) and that

\[
A = E_{1,2k+1} + \sum_{i>2k} E_{2i}a_{2i} \in L.
\]
We will show that for some basis \( \{ y_1, \ldots, y_n \} \) we have \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( F_{1,2k+1} - F_{2k+2,2} \in L \). If \( a = a_{2k+1,2} \neq 0 \) setting \( y_i = x_i \) for \( 1 \leq i \leq 2k + 1 \) and \( y_j = x_j - a^{-1}a_{2k+1,j}x_{2k+1} \) for \( j > 2k + 1 \) we see that \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( A = F_{1,2k+1} - F_{2k+1,2}a \). Then \( F_{2k+1}(\text{ad} \ A)^3 = (F_{1,2k+1} - F_{2k+1,2}a)^3a \in L \) so \( F_{1,2k+1} \in L \). This contradicts Condition \( \sim B \) so we must have \( a = 0 \). Also by Condition \( \sim B \) some \( a_{i,j} \neq 0 \). Hence we can choose \( y_{2k+2} \in \langle x_{2k+2}, \ldots, x_n \rangle \) so that \( y_{2k+2}a = -x_2 \). Then setting \( y_i = x_i \) for \( 1 \leq i \leq 2k + 1 \) and choosing \( y_j \in \langle x_{2k+2}, \ldots, x_n \rangle \cap \ker A \) for \( j > 2k + 2 \) so that \( \{ y_1, \ldots, y_n \} \) is a basis for \( V \) we have the result.

Now assume \( \text{sp}(x_1, \ldots, x_{2k}) \subseteq L \) and \( E_{1,2k+1} - E_{2k+2,2} \in L \). We will show that for some basis \( \{ y_1, \ldots, y_n \} \) of \( V \) we have \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( F_{1,2k+1} - F_{2k+2,2} \in L \), thus proving (e). We have \( E_{2k+2,2k+1} = F_{2k+1,2}((E_{1,2k+1} - E_{2k+2,2}))^2/2 \in L \). Then as in (c) we have \( B = \sum b_{ij}e_{ij} \in L \) where \( b_{ij} \in \Phi \) and satisfy \( b_{i,1+1,k+2} = -1, b_{ij} = -x_{2k+1}, y_{2k+1} = x_{2k+1}, \) and \( y_j = x_j + b_{j,k+1}x_{2k+1} \) for \( j \neq 2k+1,2k+2 \) we have \( F_{2k+1,2k+2} = -B \in L, F_{2k+2,2k+1} = E_{2k+2,2k+1} \in L \), and \( F_{1,2k+1} - F_{2k+2,2k+1} = E_{1,2k+1} - E_{2k+2,2k+1}(b_{2k+2,2k+2} - b_{2k+1,1}) \in L \). Also, for \( 1 \leq i, j \leq 2k \), we have

\[
F_{ij} = E_{ij} - E_{ij}(\text{ad} F_{2k+1,2k+2})(\text{ad} F_{2k+2,2k+1}) - F_{2k+2,2k+1}b_{ij}.
\]

Hence \( \text{sp}(y_1, \ldots, y_{2k}) \subseteq L \) and \( F_{1,2k+1} - F_{2k+2,2} \in L \) as required.

(f): If \( \dim V = 2k \) and \( \text{sp}(V) \subseteq L \) then for \( 0 \leq i \neq j \leq k - 1 \) if \( A \in L \) then

\[
E_{2i+1,2i+1}a_{2i+1,2i+1} + E_{2i+2,2i+2}a_{2i+2,2i+2} = A(\text{ad} E_{2i+1,2i+1})(\text{ad} E_{2i+2,2i+2})(\text{ad} E_{2i+1,2i+1}) \in L.
\]

Then by Condition \( \sim B \) we must have \( a_{2i+1,2i+1} + a_{2i+2,2i+2} = 0 \). Similarly we see that \( a_{2i+1,2i+2} = a_{2i+2,2i+1} = 0 \). Hence \( A = D + S \) where \( S \in \text{sp}(V) \) and \( D = \text{diag} \{ d_1, \ldots, d_{2k} \} \in L \). Now

\[
D(\text{ad}(E_{1,2} - E_{2,2})) = E_{1,2}d_1 - E_{2,2}d_2 \in L.
\]

Thus, again by Condition \( \sim B \), we have \( d_1 - d_2 = 0 \) for all \( 0 \leq i \leq k - 1 \). Thus \( 2D = I(d_1 + d_2) + E \) where \( E \in \text{sp}(V) \). Thus \( L = \text{sp}(V) \) or \( L = \text{csp}(V) \).

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