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A PRIORI BOUNDS FOR BOUNDARY SETS[1]

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ABSTRACT. Consider $y''=f(t, y, y')$ with boundary conditions $(0, y(0), y'(0))\in S_1$, $(1, y(1), y'(1))\in S_2$. It is shown that the boundary value problem has a solution for certain boundary sets S_1 and S_2 which depend on the assumed Nagumo condition for $f(t, y, y')$.

1. Consider the boundary value problem

$$y'' = f(t, y, y'), \tag{1}$$

$$(0, y(0), y'(0)) \in S_1, \qquad (1, y(1), y'(1)) \in S_2, \tag{2}$$

where $f(t, y, z)$ is continuous on $[0, 1]\times R^2$ and the boundary sets S_1 and S_2 are subsets of $[0, 1]\times R^2$ which will be defined later.

The first author together with R. Wilhelmsen has previously considered boundary value problem BVP (1)–(2) in [**1**], [**2**], and [**3**]. Jackson and Klaasen [**5**] and Sędziwy [**7**] have also investigated BVP (1)–(2). Assuming $f(t, y, z)$ satisfies a Nagumo condition, i.e., a growth restriction in the z-variable, when (t, y) lies in some compact set, a priori bounds can be determined which restrict the growth of the derivative $y'(t)$ of any solution $y(t)$ of (1) as long as $(t, y(t))$ lies in the given compact set. The purpose of this paper is to show how these a priori bounds can be utilized to determine S_1 and S_2. These a priori bounds permit the construction of boundary sets, more general than hereto considered, for which BVP (1)–(2) has at least one solution.

2. Let $\alpha(t)$, $\beta(t)\in C[0, 1]$ with $\alpha(t)\leqq\beta(t)$ for all $t\in[0, 1]$. The function $f(t, y, z)$ satisfies a Nagumo condition with respect to the pair $\alpha(t)$, $\beta(t)$ in case there exists a positive continuous function $\phi(s)$ on $[0, \infty)$ such that $|f(t, y, z)| \leqq\phi(|z|)$ for all $t\in[0, 1]$, $\alpha(t)\leqq y\leqq\beta(t)$, $|z| < \infty$, and such that

$$\int^{\infty} \frac{s}{\phi(s)}\, ds = +\infty. \tag{3}$$

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If $f(t, y, z)$ satisfies a Nagumo condition, then the following condition holds.

(A) For any $0<t_0\leqq 1$ there is $N(t_0)>0$ such that, if $y(t)$ is a solution of (1) on $[0, t_0)$ with $\alpha(t)\leqq y(t)\leqq \beta(t)$ on $[0, t_0)$, then $|y'(t)|\leqq N(t_0)$ on $[0, t_0)$. (See, for example, [**4**, p. 428].)

Condition (A) implies condition (B). A proof of this implication can be given using the Kamke convergence theorem [**4**, p. 14].

(B) Given any $B_2\geqq 0$ and any $t_0\in(0, 1]$ there exists a positive number $N(B_2)$ such that, for any solution $y(t)$ of (1) with $|y'(0)|\leqq B_2$ and $\alpha(t)\leqq y(t)\leqq\beta(t)$ for all $t\in[0, t_0]$, $|y'(t)|\leqq N(B_2)$ for all $t\in[0, t_0]$.

For $\alpha(t)$, $\beta(t)\in C[0, 1]$ with $\alpha(t)\leqq\beta(t)$ for all $t\in[0, 1]$, define $T=\{(t, y, z)\,|\, t\in[0, 1], \alpha(t)\leqq y\leqq\beta(t), |z|<\infty\}$.

For $t\in[0, 1]$, let $C(t)=\{(t, y, z)\,|\,\alpha(t)\leqq y\leqq\beta(t), |z|<\infty\}$, $S_\alpha(t)=\{(t, y, z)\,|\, y=\alpha(t), |z|<\infty\}$, and $S_\beta(t)=\{(t, y, z)\,|\, y=\beta(t), |z|<\infty\}$.

THEOREM 1. *Assume that condition* (B) *holds and that there exist* $\alpha(t)$, $\beta(t)\in C^2[0, 1]$, $\alpha(t)<\beta(t)$, *such that* $\alpha''(t)>f(t, \alpha(t), \alpha'(t))$, $\beta''(t)<f(t, \beta(t), \beta'(t))$ *on* $[0, 1]$. *If* S_1 *is a compact connected subset of* $C(0)$ *which intersects both* $\{(0, \alpha(0), z)\,|\, z\leqq\alpha'(0)\}$ *and* $\{(0, \beta(0), z)\,|\, z\geqq\beta'(0)\}$ *and if* S_2 *is a closed connected subset of* $C(1)$ *which intersects the line* $\{(1, y, z)\,|\, z=k\}$ *for every real* k, *then BVP* (1)–(2) *has a solution* $y(t)$ *with* $\alpha(t)\leqq y(t)\leqq\beta(t)$ *on* $[0, 1]$.

PROOF. Define

$$
\begin{aligned}
\bar{f}(t, y, y') &= f(t, \beta(t), y') + (y-\beta(t)), & \beta(t) &< y,\\
&= f(t, y, y'), & \alpha(t) &\leqq y\leqq\beta(t),\\
&= f(t, \alpha(t), y') + (y-\alpha(t)), & \alpha(t) &> y,
\end{aligned}
\tag{4}
$$

for all $t\in[0, 1]$, $|y'|<\infty$. It suffices to show that the boundary value problem

$$
y'' = \bar{f}(t, y, y')
\tag{5}
$$

with boundary conditions (2) has a solution $y(t)$ with $\alpha(t)\leqq y(t)\leqq\beta(t)$ on $[0, 1]$ since by the definition of $\bar{f}(t, y, y')$, $y(t)$ is then a solution of BVP (1)–(2).

Note that any solution $y(t)$ of (5) with $(0, y(0), y'(0))\in S_1$, which satisfies $(t, y(t), y'(t))\in T$ on $[0, t_0]$ for some $t_0\in(0, 1)$, and $(t_0, y(t_0), y'(t_0))\in\text{bndy } T$ has the property that $(t, y(t), y'(t))\notin T$ for any $t\in(t_0, 1]$ for which $y(t)$ exists. In addition if $(0, y(0), y'(0))\in S_1$ is such that $y(0)=\beta(0)$ and $y'(0)\geqq\beta'(0)$ or $y(0)=\alpha(0)$ and $y'(0)\leqq\alpha'(0)$, then $(t, y(t), y'(t))\notin T$ for any $t\in(0, 1]$ for which $y(t)$ exists.

For assume $t_0 \in [0, 1)$, $(t, y(t), y'(t)) \in T$ for $t \in [0, t_0]$, and, for definiteness, $y(t_0) = \beta(t_0)$. Then $y'(t_0) \geqq \beta'(t_0)$ and $\beta(t) < y(t)$ for all t in some right neighborhood of t_0. If $\beta(t_1) = y(t_1)$ for some $t_1 \in (t_0, 1]$, then there exists $t_2 \in (t_0, t_1)$ with $y(t_2) > \beta(t_2)$, $y'(t_2) = \beta'(t_2)$, and $y''(t_2) \leqq \beta''(t_2)$. However, $\beta''(t_2) < f(t_2, \beta(t_2), \beta'(t_2)) = \bar{f}(t_2, \beta(t_2), \beta'(t_2)) = \bar{f}(t_2, \beta(t_2), y'(t_2)) < \bar{f}(t_2, y(t_2), y'(t_2)) = y''(t_2)$.

The existence of a solution $y(t)$ of BVP (5)–(2), follows from [**1**, Theorem 3, p. 1061]. Hence, there exists a solution $y(t)$ of (5) satisfying (2) with $\alpha(t) \leqq y(t) \leqq \beta(t)$. By definition of $\bar{f}(t, y, z)$, $y(t)$ is a solution of BVP (1)–(2).

The next theorem is similar to those in [**3**] and [**7**]. Because a Nagumo condition is assumed, our proof is simpler.

THEOREM 2. *Assume $f(t, y, z)$ satisfies a Nagumo condition with respect to $\alpha(t)$, $\beta(t) \in C^2[0, 1]$ with $\alpha(t) \leqq \beta(t)$, $\alpha''(t) \geqq f(t, \alpha(t), \alpha'(t))$, and $\beta''(t) \leqq f(t, \beta(t), \beta'(t))$ for all $t \in [0, 1]$. If S_1 and S_2 are as in Theorem* 1, *then BVP* (1)—(2) *has a solution $y(t)$ with $\alpha(t) \leqq y(t) \leqq \beta(t)$ on* $[0, 1]$.

PROOF. Let $\bar{f}(t, y, z)$ be the modification of $f(t, y, z)$ defined by (4). Define the sequences $\{\beta_n(t)\}$ and $\{\alpha_n(t)\}$ by $\beta_n(t) = \beta(t) + 1/n$, $\alpha_n(t) = \alpha(t) - 1/n$ for all integers $n \geqq 1$ and all $t \in [0, 1]$. Note that on $[0, 1]$ each pair $\alpha_n(t)$, $\beta_n(t)$ is such that $\alpha_n(t) < \beta_n(t)$, $\alpha_n''(t) > \bar{f}(t, \alpha_n(t), \alpha_n'(t))$, and $\beta_n''(t) < \bar{f}(t, \beta_n(t), \beta_n'(t))$.

By assumption, $f(t, y, y')$ satisfies a Nagumo condition with respect to $\alpha(t)$ and $\beta(t)$. From the definition of $\bar{f}(t, y, y')$, it follows that $|\bar{f}(t, y, y')| \leqq \phi(|y'|) + 1$ for all $t \in [0, 1]$, $\alpha_n(t) \leqq y \leqq \beta_n(t)$, and $|y'| < \infty$ where ϕ is the Nagumo function for f. It is clear that

$$\int^{\infty} \frac{s}{\phi(s) + 1}\, ds = +\infty \quad \text{if} \int^{\infty} \frac{s}{\phi(s)}\, ds = +\infty.$$

Hence, $\bar{f}(t, y, y')$ satisfies a Nagumo condition with respect to every pair $\alpha_n(t)$, $\beta_n(t)$. Choose $(0, y_1, y_1') \in S_1 \cap S_\beta(0)$ and $(0, y_2, y_2') \in S_1 \cap S_\alpha(0)$ such that $y_1' \geqq \beta'(0)$ and $y_2' \leqq \alpha'(0)$ and define

$$S_1^* = S_1 \cup \{(0, y, y_1') \mid \beta(0) \leqq y \leqq \beta(0) + 1\}$$
$$\cup \{(0, y, y_2') \mid \alpha(0) - 1 \leqq y \leqq \alpha(0)\}.$$

Since each $\alpha_n(t)$ and $\beta_n(t)$ satisfies $\alpha_n''(t) > f(t, \alpha_n(t), \alpha_n'(t))$ and $\beta_n''(t) < f(t, \beta_n(t), \beta_n'(t))$, respectively, with $\alpha_n(t) < \beta_n(t)$ and since $\bar{f}(t, y, z)$ satisfies a Nagumo condition with respect to α_n and β_n (hence, condition (B) relative to $\alpha_n(t)$ and $\beta_n(t)$ is satisfied), we may evoke Theorem 1 to conclude that $y'' = \bar{f}(t, y, y')$, $(0, y(0), y'(0)) \in S_1^*$, $(1, y(1),$

$y'(1)) \in S_2$, has a solution $y_n(t)$ for each $n \geqq 1$ with $\alpha_n(t) \leqq y_n(t) \leqq \beta_n(t)$ on $[0, 1]$.

By the Arzela-Ascoli Theorem, we conclude that there exists a solution $y(t) \in C^2[0, 1]$ of BVP (5)–(2) with $\alpha(t) \leqq y(t) \leqq \beta(t)$. But then $y(t)$ is in fact a solution of BVP (1)–(2).

3. Assuming $f(t, y, z)$ satisfies a Nagumo condition with respect to $\alpha(t), \beta(t) \in C^2[0, 1]$ for $\alpha(t) \leqq \beta(t)$ for all $t \in [0, 1]$, let

$$\lambda = \max\left(\left| \alpha(0) - \beta(1) \right|, \left| \alpha(1) - \beta(0) \right|, \max_{[0,1]} \left| \alpha'(t) \right|, \max_{[0,1]} \left| \beta'(t) \right| \right)$$

and define $N(t)$ by

$$\int_\lambda^{N(t)} \frac{s}{\phi(s)} ds = \max_{u \in [0,t]} \beta(u) - \min_{u \in [0,t]} \alpha(u).$$

Let

$$F(x) = \int_\lambda^x \frac{s}{\phi(s)} ds,$$

then

$$N(t) = F^{-1}\left(\max_{[0,t]} \beta(u) - \min_{[0,t]} \alpha(u) \right)$$

is a continuous function on $[0, 1]$. Let $N = \min_{[0,1]} N(t)$. Define

$$\begin{aligned} S_3 &= \{(0, y, N) \mid \alpha(0) \leqq y \leqq \beta(0)\} \\ &\quad \cup \{(0, \beta(0), y') \mid \beta'(0) \leqq y' \leqq N\}, \\ S_4 &= \{(0, y, -N) \mid \alpha(0) \leqq y \leqq \beta(0)\} \\ &\quad \cup \{(0, \alpha(0), y') \mid -N \leqq y' \leqq \alpha'(0)\}, \\ S_5 &= \{(1, y, N(1)) \mid \alpha(1) \leqq y \leqq \beta(1)\} \\ &\quad \cup \{(1, \alpha(1), y') \mid \alpha'(1) \leqq y' \leqq N(1)\}, \\ S_6 &= \{(1, y, -N(1)) \mid \alpha(1) \leqq y \leqq \beta(1)\} \\ &\quad \cup \{(1, \beta(1), y') \mid -N(1) \leqq y' \leqq \beta'(1)\}. \end{aligned}$$

We can now state and prove our main result which shows the dependence of S_1 and S_2 on the a priori bounds.

THEOREM 3. *Assume $f(t, y, z)$ satisfies a Nagumo condition with respect to $\alpha(t), \beta(t) \in C^2[0, 1]$ with $\alpha(t) \leqq \beta(t)$ and $\alpha''(t) \geqq f(t, \alpha(t), \alpha'(t))$, $\beta''(t) \leqq f(t, \beta(t), \beta'(t))$ on $[0, 1]$. If S_1 is a closed connected subset of $C(0)$ such that $S_1 \cap S_3 \neq \emptyset$, $S_1 \cap S_4 \neq \emptyset$ and if S_2 is a closed connected subset* *of* $C(1)$ *such that* $S_2\cap S_5\neq\varnothing$ *and* $S_2\cap S_6\neq\varnothing$, *then* BVP (1)–(2) *has a solution* $y(t)$ *with* $\alpha(t)\leqq y(t)\leqq\beta(t)$.

PROOF. Choose $(0, y_1, y_1')\in S_1\cap S_3$, $(0, y_2, y_2')\in S_1\cap S_4$ such that both points belong to the same component, $\bar{S}_1$, of $S_1\cap\{(0, y, z)|\ |z|\leqq N\}$. Choose $(0, \beta(0), y_3')\in S_\beta(0)$ as follows. If $y_1=\beta(0)$, let $y_3'=y_1'$; if $y_1<\beta(0)$, choose $y_3'>N$. Let L_1 be the line segment joining $(0, y_1, y_1')$ to $(0, \beta(0), y_3')$ where, in the case when $y_1=\beta(0)$, $L_1=\{(0, y_1, y_1')\}\subset\bar{S}_1$. In a similar manner, choose $(0, \alpha(0), y_4')\in S_\alpha(0)$ by letting $y_4'=y_2'$ if $y_2=\alpha(0)$ or $y_4'<-N$ if $y_2>\alpha(0)$, and take L_2 to be the line segment joining $(0, y_2, y_2')$ to $(0, \alpha(0), y_4')$. Let $S_1^*=L_1\cup\bar{S}_1\cup L_2$ and observe that S_1^* is compact, connected and intersects both $\{(0, \alpha(0), z)|z\leqq\alpha'(0)\}$ and $\{(0,\beta(0),z)|z\geqq\beta'(0)\}$.

Pick $(1, y_5, y_5')\in S_2\cap S_5$ and $(1, y_6, y_6')\in S_2\cap S_6$ such that both points belong to the same component, $\bar{S}_2$, of $S_2\cap\{(1, y, z)|\ |z|\leqq N(1)\}$. Let L_5 and L_6 be the half-lines given by $\{(1, y_5, z)|z\geqq y_5'\}$ and $\{(1, y_6, z)|z\leqq y_6'\}$ and let $S_2^*=L_5\cup\bar{S}_2\cup L_6$.

Consider the boundary value problem $y''=f(t, y, y')$,

$$(0, y(0), y'(0)) \in S_1^*, \qquad (1, y(1), y'(1)) \in S_2^*. \tag{6}$$

By the construction of S_1^* and by the assumption on $f(t, y, z)$, the proof of Theorem 2 implies that there is a compact connected subset $C\subset C(1)$ of the funnel cross-section $F(1, 0, S_1^*)=\bigcup_{s\in S_1^*}F(1, 0, s)$ (where $F(1, 0, s)$ denotes the funnel cross-section at $t=1$ of all solutions emanating from s which exist at $t=1$) which intersects both $\{(1, y, z)\in S_\beta(1)|z\geqq\beta'(1)\}$ and $\{(1, y, z)\in S_\alpha(1)|z\leqq\alpha'(1)\}$. Hence, BVP (1)–(6) has, by Theorem 2 and the Nagumo condition, a solution $y(t)$ with $\alpha(t)\leqq y(t)\leqq\beta(t)$, $|y'(1)|\leqq N(1)$, and in fact $(1, y(1), y'(1))\in S_2$. From the Nagumo condition, $|y'(0)|\leqq N$ and by the construction of S_1^*, $(0, y(0), y'(0))\in S_1$. Therefore, BVP (1)–(2) has a solution $y(t)$.

4. As an application of these results consider the following boundary value problem previously considered by Markus and Amundson [6]. This problem arises in studying the dynamics of certain chemical reactions and is given by

$$y'' = -L_1y' - L_2k(y), \tag{7}$$

$$y(0) = 0, \qquad y'(1) = -L_1y(1). \tag{8}$$

The positive constants L_1 and L_2 depend on various parameters of the physical problem and $k(y)$ is a continuous, nonincreasing function on $[0, c]$ with $k(y)>0$ for $0\leqq y<c$ and $k(c)=0$.

Theorem 3 gives the existence of a solution $y(t)$ to BVP (7)–(8) with $0 \leqq y(t) \leqq c$ since $\alpha(t)=0$ and $\beta(t)=c$ are lower and upper solutions, respectively, and a Nagumo condition is clearly satisfied.

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