PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, No. 2, July 1971

REMARK ON DISCRETE SUBGROUPS[1]

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To A. A. Albert on his sixty-fifth birthday

ABSTRACT. One wants to know the extent to which a locally compact group G is determined by the isomorphism class of a discrete uniform subgroup Γ. Among other things, we show that if G has only finitely many components and K is a maximal compact subgroup then Γ determines the dimension of the space G/K. We then specialize our results to the case where G/K is a riemannian symmetric space.

THEOREM. *Let G_1 and G_2 be locally compact groups each with only finitely many components. Let K_i be a maximal compact subgroup in G_i, let $X_i=G_i/K_i$ coset space, and let Γ_i be a discrete subgroup of G_i.*

(1) *X_i has a unique G_i-invariant structure of finite dimensional manifold.*

(2) *If Γ_i is uniform in G_i ($i=1, 2$), and if Γ_1 and Γ_2 have subgroups of finite index that are isomorphic, then* $\dim X_1=\dim X_2$.

(3) *If* $\dim X_1=\dim X_2$, *Γ_1 is uniform in G_1, and Γ_1 and Γ_2 have subgroups of finite index that are isomorphic, then Γ_2 is uniform in G_2.*

PROOF. Let U_i be an open neighborhood of 1 in G_i and L_i maximal among the compact normal subgroups of G_i contained in U_i. Then G_i/L_i has no small subgroups, hence is a Lie group, and $X_i=(G_i/L_i)/(K_i/L_i)$. Statement (1) follows.

U_i may be chosen so that $U_i\cap\Gamma_i=\{1\}$. Then Γ_i projects isomorphically onto a discrete subgroup of the Lie group G_i/L_i. In proving (2) and (3) now we may assume that the G_i are Lie groups. We may also replace G_i by its identity component G_i^0 and Γ_i by $\Gamma_i\cap G_i^0$. We may further cut the Γ_i down to their subgroups of finite index that are isomorphic.

Now G_i is a connected Lie group, Γ_i is a discrete subgroup, and $\Gamma_1\cong\Gamma_2$.

Assume Γ_1 uniform in G_1. Then Γ_1, hence also Γ_2, is finitely generated. Let π_i denote the adjoint representation of G_i, so $\operatorname{Ker}(\pi_i)$ is the center of G_i and $A_i=\Gamma_i\cap\operatorname{Ker}(\pi_i)$ is a discrete central subgroup of G_i.

Received by the editors October 13, 1970.

AMS 1970 *subject classifications*. Primary 22E40.

Key words and phrases. Discrete subgroup, locally compact group, Lie group, symmetric space.

[1] Research partially supported by NSF Grant GP-8007.

Now A_i is a finitely generated abelian subgroup of Γ_i that is central in G_i. $\Gamma_i/A_i=\pi_i(\Gamma_i)$ is a finitely generated real linear group; so [2, Lemma 8] it has a torsion free subgroup of finite index. Cut the Γ_i down to isomorphic subgroups of finite index so that the Γ_i/A_i are torsion free. Now A_i is a finitely generated central subgroup of Γ_i that contains every torsion element. Thus the torsion elements of Γ_i form a finite abelian group Σ_i that is central in G_i. Note $\Sigma_1\cong\Sigma_2$ and $\Gamma_1/\Sigma_1\cong\Gamma_2/\Sigma_2$. Now replace G_i by G_i/Σ_i, K_i by K_i/Σ_i and Γ_i by Γ_i/Σ_i. In summary, we may assume the Γ_i torsion free.

Γ_i is discrete in G_i and torsion free, and K_i is compact; so Γ_i acts freely and properly discontinuously on $X_i=G_i/K_i$. Thus we have covering spaces $X_i\rightarrow Q_i=\Gamma_i\backslash X_i$. X_i is acyclic so $X_i\rightarrow Q_i$ is a universal Γ_i-bundle. As $\Gamma_1\cong\Gamma_2$ now Q_1 and Q_2 are homotopy equivalent. Also, the Q_i are manifolds.

Let n_i be the smallest integer such that $H^q(Q_i;\, Z_2)=0$ for $q>n_i$. Then $n_1=n_2$ because the Q_i are homotopy equivalent. Q_1 is compact because Γ_1 is uniform in G_1; thus also $n_1=\dim Q_1$.

If Γ_2 is uniform in G_2 then Q_2 is compact and $n_2=\dim Q_2$. In that case $\dim X_1=\dim Q_1=n_1=n_2=\dim Q_2=\dim X_2$. Statement (2) is proved.

If $\dim X_1=\dim X_2$ then $\dim Q_2=\dim X_2=\dim X_1=\dim Q_1=n_1=n_2$. Whenever Q is a noncompact n-manifold we know (put $q=0$, $A=Q$, $B=\emptyset$ and $G=Z_2$ in [3, Theorem 6.4]) that $H^n(Q;\, Z_2)=0$. Thus Q_2 is compact; so Γ_2 is uniform in G_2. Statement (3) is proved. □

REMARK. The following fact was seen in the proof. Let G be a locally compact group with only finitely many components and Γ a finitely generated discrete subgroup. Then Γ has a subgroup Δ of finite index, and Δ has a finite central subgroup Σ, such that Δ/Σ is torsion free.

We specialize to the semisimple case. The dimension statement in the following corollary is a special case of a forthcoming result of G. D. Mostow [1]. It is useful by itself and we supply an independent short proof.

COROLLARY. *Let G_i $(i=1, 2)$ be connected semisimple Lie groups with finite center, K_i a maximal compact subgroup of G_i, and Γ_i a discrete uniform subgroup of G_i. Suppose that Γ_1 and Γ_2 have subgroups of finite index that are isomorphic. Then the symmetric spaces $X_i=G_i/K_i$ of noncompact type satisfy*

$$\operatorname{rank} X_1 = \operatorname{rank} X_2 \quad \textit{and} \quad \dim X_1 = \dim X_2.$$

In other words, real ranks and dimensions satisfy

$$\operatorname{rank}_R G_1 = \operatorname{rank}_R G_2 \quad \textit{and} \quad \dim G_1 - \dim K_1 = \dim G_2 - \dim K_2.$$

INDEPENDENT SHORT PROOF. Passing to subgroups of finite index we may assume [2, Lemma 8] that the Γ_i are torsion free and isomorphic. As the X_i are acyclic now the $p_i: X_i \to Q_i = \Gamma_i \backslash X_i$ are universal Γ_i-bundles, and $\Gamma_1 \cong \Gamma_2$ shows the Q_i homotopy equivalent. But the Q_i are compact, so $\dim X_i = \dim Q_i$ is the smallest integer n_i such that $H^q(Q_i; Z_2) = 0$ for $q > n_i$. Thus $\dim X_1 = \dim X_2$.

We know [4, Theorem 4.2] that $r_i = \operatorname{rank}_R G_i$ is the maximum of the ranks of the free abelian subgroups of Γ_i, so $\operatorname{rank}_R G_1 = \operatorname{rank}_R G_2$. However $\operatorname{rank} X_i = \operatorname{rank}_R G_i$, so $\operatorname{rank} X_1 = \operatorname{rank} X_2$. $\square$

REFERENCES

1. G. D. Mostow, *Results announced at the International Congress of Mathematicians*, Nice, 1970 (to appear).

2. A. Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Internat. Colloq. Function Theory, Contributions to Function Theory (Bombay, 1960), Tata Inst. of Fundamental Research, Bombay, 1960, pp. 147–164. MR **24** #A188.

3. E. Spanier, *Duality in topological manifolds*, Colloque de Topologie (Brussels, 1964), Librairie Universitaire, Louvain, 1966, pp. 91–111. MR **36** #3363.

4. J. A. Wolf, *Discrete groups, symmetric spaces, and global holonomy*, Amer. J. Math. **84** (1962), 527–542. MR **26** #5523.

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