PERFECT MATRIX METHODS

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Abstract. Let \( e_i = (\delta_{ij})_{j=1}^{\infty} \), \( \Delta = (e_i)_{i=1}^{\infty} \) and let \( A \) be an infinite matrix which maps \( E \) into \( E \) where \( E \) is an FK-space with Schauder basis \( \Delta \). Necessary and sufficient conditions in terms of the matrix \( A \) are obtained for \( E \) to be dense in the summability space \( E_A = \{ x \mid Ax \in E \} \) and conditions are obtained to guarantee that \( E_A \) has Schauder basis \( \Delta \). Finally it is shown that if weak and strong sequential convergence coincide in \( E \) then in \( E_A \) the series \( \sum x_k \epsilon_k \) converges to \( x \) strongly if and only if it converges to \( x \) weakly.

1. Introduction. If \( x \) is a sequence of scalars and \( A = (a_{nk}) \) is an infinite matrix then by \( Ax \), the \( A \)-transform of \( x \), we mean the sequence \( y_n = (Ax)_n = \sum x_k \epsilon_{nk} \) provided each of these sums converge. If \( E \) is any FK-space then \( E_A \) denotes the collection of all sequences \( x \) such that \( Ax \in E \). The space \( E_A \) inherits a topology which makes it into an FK-space [5, p. 226]. A matrix \( A \) with the property that \( Ax \in E \) whenever \( x \in E \) will be called an \( E-E \) method. If \( A \) is an \( E-E \) method then \( E \subseteq E_A \); if in addition \( E = E_A \) then \( A \) is called perfect. Let \( \phi \) denote the space of all finitely nonzero sequences, \( l \) the space of absolutely summable sequences (with \( \|x\| = \sum |x_k| \)) and \( \Delta = (e_i)_{i=1}^{\infty} \), where \( e_i \) is the sequence \( (\delta_{ij})_{j=1}^{\infty} \).

In [3] it is shown that a reversible \( l-l \) method is perfect if and only if the matrix \( A \) has no nonzero left annihilators in \( m \), the space of bounded sequences. In [2] conditions are obtained for a general \( l-l \) method to be perfect. It is also shown in [2] that the series \( \sum x_k \epsilon_k \) converges strongly to \( x \in l_d \) if and only if it converges weakly to \( x \). The purpose of this note is to show that many of the results obtained in [2] and [3] for the summability field of an \( l-l \) method carry over to the summability field of an \( E-E \) method when \( E \) is an FK-space with basis \( \Delta \). In particular we show (Theorem 9) that if weak and strong sequential convergence coincide in \( E \) then for \( x \in E_A \) the series \( \sum x_k \epsilon_k \) converges to \( x \) strongly if and only if it converges weakly and (Theorem 2) that a reversible \( E-E \) method \( A \) is perfect if and only if \( A \) has no nonzero left annihilators in the sequence space representation of its dual. We will assume throughout this note that \( E \)
is an FK-space with basis $\Delta$ and so in particular every such space contains $\phi$.

2. **Notation and terminology.** An $E$-$E$ method is said to be reversible if the equation $y = Ax$ has a unique solution $x$ for each $y \in E$. If $A$ is a reversible $E$-$E$ method then $E_A$ is topologically isomorphic to $E$ under the map $A$ [5, Corollary 5, p. 204, Corollary 1, p. 199]. If the $E$-$E$ method $A$ is reversible then every $f \in E_A^*$ can be written in the form $g \circ A$ for $g \in E^*$, where $*$ denotes the space of continuous linear functionals.

If $x$ and $y$ are sequences then $(x, y)$ will denote the sum $\sum kx_ky_k$ and $xA$ denotes the sequence $(\sum x_n a_{nk})_{n=1}^\infty$. For $E$ an FK-space let $E^s = \{t | f \in E^*\}$, where $t_f = (f(e_n))_{n=1}^\infty$. Let $bs$ denote the set of sequences with finite norm $\|x\| = \sup_n |x_n| \sum_{j=1}^n x_j^j$, $cs$ the set of sequences $x$ for which $\sum_k x_k - x_{k+1}$, $c_0$ the sequences which converge to zero with the sup norm and $bv_0 = bv \cap c_0$ with the norm of $bv$. Each of the above is a BK-space. Finally we let $\omega$ denote the FK-space of all scalar sequences with the product topology.

3. **Principal results.** Motivated by the notions of type $M$, type $M^*$ (see, for example, [1, p. 90], [4, p. 184] and [2, p. 358]) and the fact that $I_E = m$ and $cs = l^1$ we make the following definition.

**Definition 1.** An $E$-$E$ method $A$ is said to be of type $E^s$ if whenever $t_A = 0$ for $t \in E^s$ then $t = 0$.

**Theorem 2.** Let $A$ be a reversible $E$-$E$ method; then $A$ is perfect if and only if $A$ is of type $E^s$.

**Proof.** $(\Leftarrow)$ It suffices to show that $\Delta$ is a fundamental set in $E_A$. Let $f \in E_A^*$ and suppose that $f(e_k) = 0$ for each $k$. Since $f \in E_A^*$ there exists a $g \in E^*$ with $f = g \circ A$. Thus $0 = f(e_k) = g[Ae_k] = g([a_{1k}, a_{2k}, \ldots])$ for each $k$. For $g \in E^*$ and $x \in E$, $g(x) = \sum n g(e_n) x_n$ and hence $\sum n g(e_n) a_{nk} = 0$ for each $k$. Since $A$ is of type $E^s$ it follows that $g(e_n) = 0$ for each $n$ and hence $g \equiv 0$. Thus for $x \in E_A$, $f(x) = g[Ax] = 0$ and so $\Delta$ is a fundamental set in $E_A$.

$(\Rightarrow)$ Assume now that $E = E_A$ and that $t_f A = 0$ for some $f \in E^*$. Let $F_a$ denote the $E_A$ topology and let $A | E$ denote $A$ considered as a linear operator from $E$ into $E$. Since $A : E_A \to E$ is continuous and $f \in E^*$ it follows that $f \circ A | E \in (E, F_a)^*$. Furthermore $\Delta$ is a basis for $(E, F_a)$ since the $F_a$ topology is weaker than the topology of $E$ [5, p. 203]. Now $f[A e_k] = f[\sum n a_{nk} e_n] = \sum n a_{nk} f(e_n) = (t_f A) k$. 

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Therefore $\phi \subseteq (f \circ A | E)^\perp$ but $(f \circ A | E)^\perp$ is $F_a$-closed in $E$ and $\phi$ is $F_a$-fundamental in $E$, hence $f \circ A | E \equiv 0$. The zero functional and $f \circ A$ are both continuous extensions of $f \circ A | E$ to all of $E_A$. Since $E = E_A$ it follows that $f \circ A \equiv 0$ and hence by the reversibility of $A$, $f \equiv 0$. Thus $f = 0$ and $A$ is of type $E'$.

Since $l^b = m$ we obtain as a corollary the following theorem of Brown and Cowling [3, Theorem 2].

**Corollary 3.** A reversible $l$-$l$ method is perfect if and only if it is of type $M^\ast$.

Similarly for reversible $E$-$E$ methods, where $E$ is one of the familiar sequence spaces $c_0$, $c_0$ or $bv_0$, we have that perfectness is equivalent to type $bo$, type $l$, and type $bs$ respectively.

**Definition 4.** If $A$ is an $E$-$E$ method and $t \in E^b$ we say that $t$ has property $P$ if $(tA, x) = \sum^\infty_{k=1} \sum a_{nk} x_k$ converges for each $x \in E_A$. The set of all $t \in E^b$ with property $P$ is denoted by $Q$. The method is called associative if $Q = E^b$ and $f[Ax] = (t_fA, x)$ for each $f \in E^*$ and each $x \in E_A$ (cf. [2, p. 282]).

**Lemma 5.** Let $A$ be an $E$-$E$ method and let $t \in Q$ then $(tA, \cdot)$ defines a continuous linear functional on $E_A$.

**Proof.** Let $g_j = \sum^\infty_{k=1} (\sum a_{nk} x_k) E_k$ and $g(x) = (tA, x)$, where $E_k$ is the $k$th coordinate functional. Since $E_A$ is an FK-space $g_j \in E^*_A$ for each $j$ and since $t \in Q$, $g_j \to g$ pointwise on $E_A$. The continuity of $g$ follows from [5, p. 200].

**Theorem 6.** Let $A$ be an $E$-$E$ method. Then $A$ is perfect if and only if $f[Ax] = (t_fA, x)$ for each $x \in E_A$ and each $t_f \in Q$ (cf. [3, Theorem 1] and [2, Theorem A]).

**Proof.** $(\Rightarrow)$ Let $t_f \in Q$ and let $g(x) = (t_fA, x)$ for $x \in E_A$; then $f[Ax] = f[\sum a_{nk} x_k] = (t_fA, e_k)$ and so $g = f \circ A$ on the fundamental set $\Delta$. Since $g$ and $f \circ A$ are continuous on $E_A$ it follows that $g = f \circ A$.

$(\Leftarrow)$ Let $f \in E^*_A$ be such that $f(e_k) = 0$ for each $k$. By [5, p. 230] there exists $F \in \omega_A^*$ and $G \in E^*$ such that $f(x) = F(x) + G[Ax]$ for each $x \in E_A$. Therefore $0 = f(e_k) = F(e_k) + G[\sum a_{nk} e_n] = F(e_k) + \sum a_{nk} G(e_n)$. Since $\Delta$ is a basis for $\omega_A$ [5, p. 230] we have in particular that $F(x) = \sum F(e_k) x_k$ for each $x \in E_A$. Combining these results we have that

$$\sum F(e_k) x_k = - \sum_k \left( \sum_n G(e_n) a_{nk} \right) x_k$$

for each $x \in E_A$. Thus
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\[ f(x) = F(x) + G[Ax] \]

\[ = \sum_k F(e_k)x_k + \sum_n G(e_n) \sum_k a_{nk}x_k \]

\[ = \sum_n G(e_n) \sum_k a_{nk}x_k - \sum_k \left( \sum_n G(e_n)a_{nk} \right)x_k \]

\[ = G[Ax] - (t_A, x) = 0. \]

Hence \( f = 0 \) and so \( E = E_A \).

**Theorem 7.** Let \( A \) be an \( E-E \) method. Then \( A \) is associative if and only if \( E_A \) has basis \( \Delta \).

**Proof.** \((\imp)\) Let \( x \in E_A \) and \( f \in E_A^* \). Choose \( F \in \omega_A^* \), \( G \in E^* \) such that \( f = F + G \circ A \) and let \( y_n = x - \sum_{k=1}^n x_ek_{ek} \). Then

\[ f(y_n) = F(y_n) + G[Ay_n] = F(y_n) + (t_A, y_n) \]

\[ = F(y_n) + \sum_{k=1}^\infty \left( \sum_{j=1}^\infty G(e_j)a_{jk} \right)x_k. \]

The first term \( \lim n \)'s to 0 on \( n \) since \( \Delta \) is a basis for \( \omega_A \) and the second limits to 0 since the double series converges. Thus \( \Delta \) is a weak basis and hence a basis for \( E_A \).

\((\lim)\) Let \( x \in E_A \) and \( f \in E^* \) then \( f \circ A \in E_A^* \) and so

\[ f[Ax] = \sum_k x_kf[Ae_k] = \sum_k x_k \sum_n g_{nk}(e_n) = (t_A, x). \]

We shall say that \( x \in E_A \) is perfect if \( f(Ax) = (t_A, x) \) for each \( t_A \in Q \) and that \( x \) is associative if \( Q = E^* \) and \( f(Ax) = (t_A, x) \) for all \( t_A \in Q \).

**Theorem 8.** Let \( A \) be an \( E-E \) method and let \( x \in E_A \); then

(i) \( \sum_k x_k e_k \) converges to \( x \) weakly if and only if \( x \) is associative,

(ii) \( x \) is in the closure of \( \phi \) in \( E_A \) if and only if \( x \) is perfect.

**Proof.** \((\imp)\) Let \( t_A \in E^* \) and let \( F = f \circ A \); then \( F \in E_A^* \) and

\[ F(x) = \sum_k x_k F(e_k) = \sum_k x_k f(Ae_k) = \sum_k x_k \sum_n a_{nk}(e_n) = (t_A, x). \]

\((\lim)\) Let \( g \in E_A^* \); say \( g = F + G \circ A \) for \( F \in \omega_A^* \) and \( G \in E^* \); then\n
\[ g(e_k) = F(e_k) + \sum_n G(e_n)a_{nk}. \]

Thus

\[ g(x) = F(x) + G[Ax] = \sum_k x_k F(e_k) + G[Ax] \]

\[ = \sum_k x_k \left( g(e_k) - \sum_n G(e_n)a_{nk} \right) + G[Ax] \]

\[ = \sum_k g(e_k) - (t_A, x) + G[Ax] = \sum_k x_k g(e_k). \]
(ii) \( \Rightarrow \) Let \( x \) be in the closure of \( \phi \) in \( E_A \) and let \( t \in \mathbb{Q} \). Define \( g: E_A \to \mathbb{R} \) by \( g(y) = f[Ay] - (t \circ A, y) \); then \( g \in E_A^* \) by Lemma 5 but \( g(e_k) = 0 \) for each \( k \) and so \( g(x) = 0 \). Therefore \( f[Ax] = (t \circ A, x) \).

\( \Leftarrow \) Let \( f \in E_A^* \) be such that \( f[\phi] = 0 \). Then, as in (i), \( f(x) = \sum_k f(e_k)x_k + G[Ax] - (t \circ A, x) = G[Ax] - (t \circ A, x) \). Thus \( t \in \mathbb{Q} \) and so \( f(x) = 0 \).

For the following theorem we do not assume \( E \) has basis \( \Delta \).

**Theorem 9.** Let \( A \) be an \( E-E \) method and suppose that weak and strong sequential convergence coincide in \( E \). Then for \( x \in E_A \) the series \( \sum x_k e_k \) converges to \( x \) if and only if it converges to \( x \) weakly.

**Proof.** Let \( x \in E_A \) be such that \( \sum x_k e_k \to x \) weakly and let \( y_j = (0, \ldots, 0, x_j, x_{j+1}, \ldots) \). Let \( (r_n) \) be the determining seminorms for \( E \); then the topology of \( E_A \) is given by the seminorms \( \{E_n\} \), \( (p_n), (q_n) \), where \( q_n = r_n \circ A \) and \( p_n \) is defined by

\[
p_n(x) = \sup \left| \sum_{k=1}^{m} a_{nk} x_k \right| \quad [5, p. 226, Theorem 1].
\]

Since \( E_n \subseteq E_A^* \) for each \( n \) it is clear that \( |E_n(y_j)| \to 0 \) for each \( n \). Let \( f \in E^* \) then \( f \circ A \in E_A^* \) and hence \( f \circ A(y_j) \to 0 \). Thus \( (A(y_j)) \) converges to zero weakly and hence strongly in \( E \) and so \( q_n(y_j) \to 0 \) for each \( n \). Finally fix \( n \) and let \( \epsilon > 0 \) be given. Choose \( N \) such that \( j, m \geq N \) implies

\[
\left| \sum_{k=1}^{m} a_{nk} x_k \right| < \epsilon.
\]

Thus

\[
\sup_{m > j} \left| \sum_{k=j}^{m} a_{nk} x_k \right| \leq \epsilon \quad \text{for } j > N,
\]

but \( p_n(y_j) = \sup_{m \geq j} \left| \sum_{k=j}^{m} a_{nk} x_k \right| \) and hence \( p_n(y_j) \to 0 \) for each \( n \). It follows that \( y \to 0 \) in \( E_A \).

**Remarks.** (i) It has been pointed out by G. Bennett that the proof of Lemma 3 on p. 285 of [2] makes incorrect use of Satz 3.4 of [6, p. 60]. Since weak and strong sequential convergence coincide in \( l \) Lemma 3 of [2] follows from Theorem 9 above.

(ii) If \( E \) is an FK-space with determining seminorms \( (r_n) \) and if \( A \) is a row finite \( E-E \) method then the seminorms \( \{E_n\} \) and \( (r_n \circ A) \) are sufficient to determine the topology of \( E_A \). Thus if weak and strong sequential convergence coincide in \( E \) one can proceed as in the proof of Theorem 9 to show they coincide in \( E_A \). This result has been observed by Bennett in [7].
REFERENCES


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