ON A SEQUENCE OF ALMOST DETERMINISTIC PAIRWISE
INDEPENDENT RANDOM VARIABLES

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Abstract. We show the existence of infinite sequences of pair-
wise independent random variables (which are not constant almost
everywhere) such that the whole sequence is determined with
probability one by the knowledge of any pair of members of that se-
quence.

1. Introduction. We want to show the existence of infinite se-
quences $Y_1, Y_2, \ldots, Y_n, \ldots$ of pairwise independent random vari-
ables (which are not constant almost everywhere) having the prop-
erty that for any $i \neq j$ there exist Borel functions $g_{i,j,n}$ such that

\[ Y_n = g_{i,j,n}(Y_i, Y_j), \quad n = 1, 2, 3, \ldots. \]

This answers a question raised by D. Dawson. For finite sets, such
a construction is done in [3].

2. Construction. We will first construct a sequence of random ele-
ments having the properties mentioned in the introduction. Let $G$
be the character group of $Q$, the additive group of the rational numbers
with the discrete topology. $G$ is an abelian compact group which is
divisible and torsion free. Moreover $G$ is metrizable, complete and
separable. There is a lot of information in the literature about $G$, but
we refer only to [2] where one can find other references. $G$ being comp-
act, the Haar measure $\mu$ is uniquely determined on the Borel sets of
$G$ by $\mu(G) = 1$. Take as probability space $\Omega = G \times G$ the direct product
of the group $G$ with itself, and for measure on $\Omega$ take the Haar mea-
ure $\mu \times \mu$ which is the product measure of $\mu$ with itself. Let $X_1$ (resp.
$X_2$) denote the projection maps of $G \times G$ onto the first (resp. second)
coordinates. $X_1$ and $X_2$ are random independent elements taking
values in $G$. Define

\[ X_n = (n - 2)X_1 + X_2, \quad n = 2, 3, \ldots. \]

Given any $i \neq j$ it is easily verified, since $G$ is a vector space over the
rationals, that the relations given by (1) define $X_1$ and $X_2$ uniquely in
terms of $X_i$ and $X_j$; moreover if we write $(X_1, X_2) = g_{i,j}(X_i, X_j)$, $g_{i,j}$
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is an automorphism of $\Omega$. Let us compute the image $\bar{\mu}$ of $\mu$ under this automorphism. For any set $A \times B$ of $\Omega$ where $A$ and $B$ are Borel sets of $G$ we have

$$\bar{\mu}(A \times B) = \mu \times \mu\{(X_i, X_j) \in A \times B\} = \mu \times \mu\{(X_i, X_j) \in g_{i,j}^{-1}(A \times B)\}.$$ 

Since $g_{i,j}$ is a group isomorphism it follows that $\bar{\mu}(A \times B)$ is translation-invariant and therefore $\bar{\mu}$ is a Haar measure; since $g$ is onto $\bar{\mu}(G \times G) = 1$ and $\bar{\mu} = \mu \times \mu$. It follows that

$$\mu \times \mu\{(X_i, X_j) \in A \times B\} = \mu(X_i \in A) \cdot \mu(X_j \in B) = \mu(A) \cdot \mu(B).$$

Our sequence $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of random elements having the properties stated in the introduction. To establish the existence of random variables having these properties, it suffices to note that $G$ being complete and separable there is a Borel isomorphism $\phi$ of $G$ on $[0, 1]$ where the image of $\mu$ is Lebesgue measure. See for instance [1, p. 339]. If we define $Y_n = \phi(X_n)$ we have established the following

**Theorem.** The sequence $Y_1, \ldots, Y_n, \ldots$ defines pairwise independent random variables. Each of the $Y_i$ is uniformly distributed on $[0, 1]$ and for each $i, j, i \neq j$, there are Borel functions $g_{i,j,n}$ such that $Y_n = g_{i,j,n}(Y_i, Y_j)$, $n = 1, 2, \ldots$.

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**References**


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1 Located for the time being at the “Auberge Joffe”, Outremont, Canada.