NONSTABLE HOMOTOPY GROUPS OF THOM COMPLEXES

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Abstract. The first nonstable homotopy group of $MO(n)$, $\pi_{2n}(MO(n))$, is computed for all $n$, together with the corresponding Postnikov invariant. The computations give a new proof of a theorem of Mahowald on the normal bundle of an imbedding.

1. In 1954 R. Thom introduced the complexes $MO(n)$ in his study of the global properties of differentiable manifolds: e.g., the realization of homology classes by submanifolds and cobordism, [1]. One of his central results is the computation of $\pi_{n+h}(MO(n))$ for $h<n$. Let $d(h)$ be the number of partitions, $\{a_1, \ldots, a_r\}$, of the integer $h$ with no $a_i$ of the form $2^r-1$. (Thom calls these nondyadic partitions $\delta(1)=0$, $\delta(2)=1$, $\delta(3)=0$, $\delta(4)=2$, etc.) Then $\pi_{n+h}(MO(n))$ is the direct sum of $d(h)$ copies of $\mathbb{Z}_2$, which I abbreviate: $d(h)\cdot \mathbb{Z}_2$. Using the same methods as [1], I shall prove:

**Theorem 1.** For $n$ an odd integer, $n \neq 2^r-1$, $\pi_{2n}(MO(n)) = (d(n)+1)\cdot \mathbb{Z}_2$; for $n = 2^r-1$, $\pi_{2n}(MO(n)) = d(n)\cdot \mathbb{Z}_2$.

**Theorem 2.** For $n$ an even integer, $n \neq 2^r$, $\pi_{2n}(MO(n)) = \mathbb{Z} \oplus d(n)\cdot \mathbb{Z}_2$; for $n = 2^r$, $\pi_{2n}(MO(n)) = \mathbb{Z} \oplus (d(n)-1)\cdot \mathbb{Z}_2$.

The proofs contain somewhat more than the theorems state: the relevant Postnikov invariants are also obtained. The following section is concerned with the basic construction, information about $MO(n)$, and some technical facts about the Pontrjagin square. Theorem 1 is proven in §3 and Theorem 2 in §4. The last paragraph contains an application to manifolds.

2. I recall that $MO(n)$ is obtained from the universal disk bundle over $BO(n)$ by collapsing the entire boundary (i.e., the corresponding sphere bundle) to a point. $H^*(MO(n); \mathbb{Z}_2)$ can be identified, via the Thom isomorphism, $\varphi^*$, with the ideal generated by $w_n$ in $H^*(BO(n); \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_n]$. For $n$ an odd integer and $p$ an odd prime, $\hat{B}^*(MO(n); \mathbb{Z}_p)$ is trivial; for $n$ even, $H^{2n+4j}(MO(n); \mathbb{Z}_p) = \mathbb{Z}_p$. Thom’s computation of the stable homotopy of $MO(n)$ proceeds as
follows: (cf. pp. 36-42 of [1]). For each nondyadic partition $\omega_h = \{a_1, \ldots, a_r\}$ of $h$ a distinguished element $X_{\omega_h}$ in $H^{*+h}(MO(n); Z_2)$ is defined; each $X_{\omega_h}$ has associated with it a function $f_{\omega_h}: MO(n) \to K(Z_2, n+h)$. The map

$$ F_{2n-1} = \prod_{h<n} f_{\omega_h}: MO(n) \to \prod_{h<n} (K(Z_2, n+h))^{d(h)} = K(2n-1) $$

induces, for all coefficient fields, $Z_p$, cohomology isomorphisms in dimensions less than $2n$ and an injective homomorphism in dimension $2n$. Since the spaces $K(Z_2, n+h)$ have no $Z_p$ cohomology for $p$ an odd prime, the only nontrivial case is $p = 2$. The proof for $Z_2$ coefficients is based upon the following:

**Lemma 1.** Let $I = (i_1, \ldots, i_r)$ be an admissible sequence (i.e., $i_k \geq 2i_{k+1}$ and $2i_1 - \sum_{k=1}^{r} i_k < n$); then the elements $Sq^{i_1} \cdots Sq^{i_r} X_{\omega_h} = Sq^i X_{\omega_h}$ with $i_1 \leq n$ are linearly independent in $H^*(MO(n); Z_2)$.

Subject to the restriction that $\sum_{k=1}^{r} i_k < n$ this is proven by Thom (pp. 38-42 of [1]); the proof, in fact, carries through without significant change in the more general case, $i_1 \leq n$. Now, a theorem of Serre says that the elements $Sq^i \epsilon$, where $I$ is admissible and $\epsilon$ is the fundamental class in dimension $k$, form a basis for the algebra $H^*(Z_2, k; Z_2)$, [2]. Thus $F_{2n-1}^*: H^{*+h}(K(2n-1); Z_2) \to H^{*+h}(MO(n); Z_2)$ is injective for $h \leq n$ and it is not hard to show that for $h < n$ it is also surjective. This means that $F_{2n-1}$ gives isomorphisms in the same dimensions for all fields, $Z_p$, and by the standard J. H. C. Whitehead-Hurewicz argument, $\pi_{n+h}(MO(n)) = \pi_{n+h}(K(2n-1)) = d(h) \cdot Z_2$. Since $\pi_1(K(2n-1)) = 0$ for $i \geq 2n$, $K(2n-1)$ is the $(2n-1)$st stage of the Postnikov tower for $MO(n)$.

It is important to note that the elements $X_{\omega_h}$ are not, in general, monomials in the ideal $(w_n)$ of $H^*(BO(n); Z_2)$. For $k \geq 1$, however, $\{2^k\}$ is a nondyadic partition of $h = 2^k$ and the corresponding $X_{\omega_h}$ is $w_n \cdot w_{2^k}^*$. (This is clear when one refers to Thom’s definition of the $X_{\omega_h}$.) The classes $w_n \cdot w_{2^k}^*$ are of some importance in the next two sections; to simplify the notation I shall denote $X_{\{2^k\}}$ by $X_k$, the corresponding map of $MO(n)$ to $K(Z_2, n+2^k)$ by $f_k$, and the generator of the corresponding $H^{n+2^k}(Z_2, n+2^k; Z_2)$ by $\epsilon_k$. Thus for $k > 0$, $f_k(\epsilon_k) = X_k = w_n \cdot w_{2^k}^*$. It is also convenient to define $X_0$ to be $w_n$, the corresponding map of $MO(n)$ to $K(Z_2, n)$ to be $f_0$, and the generator of $H^n(Z_2, n; Z_2)$ to be $\epsilon_0$.

In §§4 and 5, I shall require some of the basic properties of the Pontrjagin square, $p$ [3], [4]. This is a cohomology operation of type $(n, Z_2; 2n, Z_4)$ and it is defined as follows. Let $\xi$ be a cochain represent-
ing the class \( z \) in \( H^n(X; \mathbb{Z}_2) \); \( \delta \chi = 2\eta \) for some \((n+1)\) cochain \( \eta \); \( p(z) \) is the class defined by the mod 4 cocycle \( \xi \cup \delta \xi + \chi \cup 1 \delta \chi \). Let

\[
\cdots \longrightarrow S^1 \longrightarrow H^i(X; \mathbb{Z}_2) \overset{\mu_{4,2}}{\longrightarrow} H^i(X; \mathbb{Z}_4) \overset{\beta_4}{\longrightarrow} H^i(X; \mathbb{Z}_2) \longrightarrow \cdots
\]

be the coefficient sequence derived from \( 0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0 \). It is not hard to show:

\[
\begin{align*}
(1) & \quad \mu_{4,2}(z^2) = 2p(z), \\
(2) & \quad (p(z))^2 = z^2, \\
(3) & \quad (\beta_4(p(z)))^2 = zS^1z + S^1z^2 \quad (n \text{ even}) \\
& \quad = S^1z^2 \quad (n \text{ odd}).
\end{align*}
\]

Here \( \beta_4 \) is the Bockstein operator associated with the sequence \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_4 \rightarrow 0 \) and \( (\ )^2 \) indicates reduction mod two. If \( n \) is even, \( H^{2k+1}(Z_2, k; \mathbb{Z}) \) contains a \( \mathbb{Z}_4 \) summand which is generated by \( \beta_4(p(e)) \).

\[(\text{Proof: } (\beta_4(p(e)))^2 = e - S^1z + S^1z^2; \text{ this last term is not of the form } S^1x \text{ for some } x \text{ in } H^{2k}(Z_2, k; \mathbb{Z}), \text{ so } \beta_4(p(e)) \text{ is not of order two. It is of order } \leq 4 \text{ because it is in the image of } \beta_4.\]

3. In this section \( n \) is an odd integer, \( 2^r < n < 2^{r+1} \). The 2\( n \)th stage of the Postnikov tower for \( MO(n) \) is a fibration \( p_{2n}: K_{2n} \rightarrow K_{2n-1} \) with fibre \( K(\pi_{2n}(MO(n)), 2n) \) and a map \( F_{2n}: MO(n) \rightarrow K_{2n} \) which induces homotopy isomorphisms in dimensions \( \leq 2n \). Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
F_{2n} & \downarrow & K_{2n} \\
& \rightarrow & \\
MO(n) & \downarrow & P_{2n} \\
& \rightarrow & \\
F_{2n-1} & \rightarrow & K_{2n-1}
\end{array}
\]

As I noted above, to require that \( F_{2n} \) induce homotopy isomorphisms in the first \( 2n \) dimensions is equivalent to requiring that, for all coefficient fields \( \mathbb{Z}_p \), \( F_{2n}^p \) be a cohomology isomorphism in dimensions \( \leq 2n \) and a monomorphism in dimension \( 2n+1 \). The first step in the construction of \( K_{2n} \) is to obtain the cohomology isomorphism in dimension \( 2n \). Let \( \overline{K} = K_{2n-1} \times (K(Z_2, 2n))^d(n) \) and let \( \overline{F}: MO(n) \rightarrow \overline{K} \) be the product of \( F_{2n-1} \) and the functions \( f_{\alpha e} \) associated with the distinguished classes \( X_{\alpha e} \). (There are \( d(n) \) of these classes and maps.) Both \( MO(n) \) and \( \overline{K} \) have trivial \( \mathbb{Z}_p \) cohomology for \( p \) an odd prime.
(since \( n \) is odd) and Thom has shown that \( \overline{F}^* \) is, in fact, an isomorphism in \( \mathbb{Z}_2 \) cohomology in dimension \( 2n \). It remains now to determine the kernel of \( \overline{F}^* \) in dimension \( 2n+1 \) and to eliminate it by taking the appropriate fibration over \( K \). \( \mathbb{Z}_2 \) coefficients are to be understood in the following

**Lemma 2.** If \( n = 2^{r+1} - 1 \), \( \overline{F}^* \) is a monomorphism in dimension \( 2n+1 \). If \( n \neq 2^{r+1} - 1 \), \( \text{Ker } \overline{F}^* = \mathbb{Z}_2 \) in dimension \( 2n+1 \); the nonzero element is \( \theta = \varepsilon_0 \cdot Sq^1 \varepsilon_0 + Sq^n \varepsilon_0 + \cdots + Sq^{n-1} \varepsilon_1 + \cdots + Sq^{n-2r+1} \varepsilon_r \).

**Proof of the Lemma.** I first note that \( H^{2n+1}(K; \mathbb{Z}_2) \) is the sum of the cohomology of the factors, since \( K = K(Z_2, n) \times K(Z_2, n+2) \times \cdots \times (K(Z_2, 2n))^{d(n)} \), and the cross product terms of the Künneth formula (e.g. \( \varepsilon_0 \cdot \varepsilon_1 \)) do not appear until dimension \( 2n+2 \). Thus the elements of \( H^{2n+1}(K; \mathbb{Z}_2) \) are of two kinds: those of the form \( \varepsilon_0 \cdot Sq^i \varepsilon_0 + \sum Sq^j \varepsilon_{a_j} \) and those of the form \( \sum Sq^i \varepsilon_{a_i} \). \( \text{Ker } \overline{F}^* \) cannot contain elements of the second kind, for \( \overline{F}^*(Sq^i \varepsilon_{a_i}) = Sq^i(X_{a_i}) \) and these terms are linearly independent. Moreover, if there is an element of the first kind in \( \text{Ker } \overline{F}^* \) it is unique. (Proof: Let \( \theta_1 \) and \( \theta_2 \) be two such elements. Then \( \theta_1 + \theta_2 \) is also in the kernel and it is of the second kind, since the \( \varepsilon_0 \cdot Sq^i \varepsilon_0 \) terms cancel. Hence \( \theta_1 + \theta_2 = 0 \).)

Now consider \( \theta = \varepsilon_0 \cdot Sq^1 \varepsilon_0 + Sq^n \varepsilon_0 + \cdots + Sq^{n-1} \varepsilon_1 + \cdots + Sq^{n-2r+1} \varepsilon_r \). If \( n \neq 2^{r+1} - 1 \),

\[
\overline{F}^*(\theta) = w_n \cdot w_1 + w_n \cdot w_1 + \cdots + w_n \cdot w_1 = 0,
\]

and the lemma is proven. If \( n = 2^{r+1} - 1 \), \( \overline{F}^*(\theta) = w_n \cdot w_1^{2^{r+1}} = X_{r+1} \). Suppose that \( \theta \) is in \( \text{Ker } \overline{F}^* \). Then \( \theta_1 + \theta \) is an element of the second kind, i.e., \( \theta_1 + \theta = \sum Sq^i \varepsilon_{a_i} \), so \( X_{r+1} = \overline{F}^*(\theta_1 + \theta) = \overline{F}^*(\sum Sq^i \varepsilon_{a_i}) = \sum Sq^i X_{a_i} \). But this would be a relation among the elements which Lemma 1 states are independent. Hence in this case \( F^* \) is a monomorphism.

**Proof of Theorem 1.** Let \( n = 2^{r+1} - 1 \); the space \( K \) and the map \( \overline{F} \) satisfy the requirements for the \( 2n \)th stage of the Postnikov tower. Since \( K = K_{2n-1} \times (K(Z_2, 2n))^{d(n)} \), \( \pi_{2n}(MO(n)) = d(n) \cdot \mathbb{Z}_2 \) and the \( k \)-invariant is zero.

Now suppose \( n \neq 2^{r+1} - 1 \). The cohomology class \( \theta \) corresponds to a map \( f_\bullet : \overline{K} \to K(Z_2, 2n+1) \); let \( K_{2n} \) be the fibre space over \( K \) induced by \( f_\bullet \) from the standard path fibration over \( K(Z_2, 2n+1) \). The fibre space \( K_{2n} \) is nontrivial only over the spaces \( K(Z_2, n+2k) \) which correspond to the classes \( \varepsilon_k \), \( k \leq r \). Since \( 2^r < n \), these spaces \( K(Z_2, n+2k) \) have trivial homotopy in dimension \( 2n \). Thus the homo-
topology sequence of the fibration $p_{2n}: K_{2n} \to K$ gives $\pi_{2n}(K_{2n})$ as the trivial extension of $\pi_{2n}(K) = d(n) \cdot \mathbb{Z}_2$ by $\pi_{2n}(K(Z_2, 2n)) = \mathbb{Z}_2$. Therefore $\pi_{2n}(K_{2n}) = (d(n) + 1) \cdot \mathbb{Z}_2$. Since $\overline{F}^*(\theta) = 0$, the map $\overline{F}$ can be lifted to a map $F_{2n}: MO(n) \to K_{2n}$. $F_{2n}$ induces homotopy isomorphisms in dimensions $\leq 2n$, so $\pi_{2n}(MO(n)) = (d(n) + 1) \cdot \mathbb{Z}_2$. The $k$-invariant is the characteristic class of the fibration $p_{2n}: K_{2n} \to K_{2n-1}$, which is just the class $\theta$. This completes the proof of Theorem 1.

4. In this section $n$ is an even integer, $2r \leq n < 2^{r+1}$. The procedure is the same as in §3, but the situation is made slightly more complicated by the presence of an infinite cyclic summand in $H^{2n}(MO(n); \mathbb{Z})$. (The presence of this summand is indicated by the fact that $H^{2n}(MO(n); \mathbb{Z}_p) = \mathbb{Z}_p$ for all odd primes.) The first step is to consider $\overline{F}: MO(n) \to K = K_{2n-1} \times (K(Z_2, 2n))^{d(n)}$. $\overline{F}^*$ is an isomorphism in $\mathbb{Z}_2$ cohomology in dimension $2n$, so $\pi_{2n}(MO(n)) = (d(n) + 1) \cdot \mathbb{Z}_2$. The $\beta$-invariant is the characteristic class of the fibration $p_{2n}: K_{2n} \to K_{2n-1}$, which is just the class $\theta$. This completes the proof of Theorem 1.

**Lemma 2'**. In $\mathbb{Z}_2$ cohomology in dimension $2n + 1$, $\text{Ker } \overline{F}^* = \mathbb{Z}_2$; the generator is $\theta = e_0 \cdot Sq^1 e_0 + Sq^n e_0 + Sq^{n-1} e_1 + \cdots + Sq^{n-2^r + 1} e_r$.

The proof is precisely the same as that of Lemma 2.

Since $n$ is even, $Sq^{n-1} e_1 + Sq^{n-2^r} e_2 + \cdots + Sq^0 e_r$ is in the image of $Sq^1$, so it is the reduction of some integral class, $\xi$, of order two. Now let $\mu = \beta_4(p(\epsilon_0)) + \xi^2$; $\mu$ is an integral class of order four (since $\beta_4(p(\epsilon_0))$ is) and $\mu_2 = e_0 \cdot Sq^1 e_0 + Sq^n e_0 + Sq^0 e_r + \xi^2 = \theta$. The class $\mu$ determines a mapping $f_\mu: K \to K(Z, 2n + 1)$; let $K_{2n}$ be the fibre space over $K$ induced by $f_\mu$ from the path fibration over $K(Z, 2n + 1)$. $(\overline{F}^*(\mu_2) = \overline{F}^*(\mu)) = \overline{F}^*(\theta) = 0$. Since the integral cohomology of $MO(n)$ in dimension $2n + 1$ has only elements of order two, this means that $\overline{F}^*(\mu) = 0$. Thus $\overline{F}$ can be lifted to a map $F_{2n}$ of $MO(n)$ to $K_{2n}$. It is clear that $F_{2n}$ induces the appropriate isomorphisms and monomorphisms in cohomology and that $K_{2n}$ is the $2n$th stage of the Postnikov system. It remains to compute $\pi_{2n}(K_{2n})$. The fibration $p_{2n}: K_{2n} \to K$ is nontrivial over the factors $K(Z_2, n + 2^r)$ corresponding to the classes $\epsilon_r$, $k \leq r$. Thus if $n \neq 2^r$ the homotopy sequence of the fibration $p_{2n}: K_{2n} \to K$ has the trivial extension of $\pi_{2n}(K)$ by $\pi_{2n}(K(Z, 2n))$ and so $\pi_{2n}(K_{2n}) = \mathbb{Z} \oplus d(n) \mathbb{Z}_2$. (This is equivalent to saying that the fibre of $p_{2n}: K_{2n} \to K_{2n-1}$ is $K(Z, 2n) \times (K(Z_2, 2n))^{d(n)}$.) If $n = 2^r$, the fibration over the factor $K(Z_2, 2n)$ which corresponds to the class $\epsilon_r$ is nontrivial, since $Sq^1 e_r$ is a summand of $\theta$. Thus the homotopy sequence of the fibration $K_{2n} \to K$ contains the nontrivial extension $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$, and $\pi_{2n}(K_{2n}) = \mathbb{Z} \oplus (d(n) - 1) \mathbb{Z}_2$. This is equivalent to the fact that the fibre of $p_{2n}: K_{2n} \to K_{2n-1}$ is $K(Z, 2n) \times (K(Z_2, 2n))^{d(n) - 1}$.

If $n \neq 2^r$, the Postnikov invariant in dimension $2n + 1$ is $\mu$; if $n = 2^r$,
the invariant, which is the characteristic class of the fibration $p_{2n} : K_{2n} \to K_{2n-1}$, is $2\mu = 2\beta p(e_0)$. This completes the proof of Theorem 2. I remark that this computation was made by Thom in the case $n = 2$.

5. The preceding computations give a short proof of a theorem of Mahowald [4].

**Theorem 3.** Let $M^n$ be a manifold of dimension $n = 2k$ with $w_{n-1} \cdot w_1 \neq 0$. For any imbedding of $M^n$ in $\mathbb{R}^{2n}$ with normal bundle $\nu$, the Euler characteristic $\chi(\nu)$ is congruent to two modulo four.

**Proof.** Let $f : M(\nu) \to MO(n)$ be the map of Thom complexes induced from the classifying map for $\nu$, and let $\chi$ and $\chi(\nu)$ be the integral Euler classes of $MO(n)$ and $M(\nu)$. (That is to say, $\chi$ and $\chi(\nu)$ are the images under the Thom isomorphism of the Euler classes of the respective bundles.) Consider the following portions of the coefficient sequences of $MO(n)$ and $M(\nu)$ derived from the sequence $0 \to Z_2 \to Z_4 \to Z_2 \to 0$.

\[ \cdots \to H^{2n}(MO(n); \mathbb{Z}_2) \xrightarrow{\mu_{4,2}} H^{2n}(MO(n); \mathbb{Z}_4) \xrightarrow{\rho_1} H^{2n}(MO(n); \mathbb{Z}_2) \xrightarrow{\rho_2} \cdots \]

\[ \cdots \to H^{2n}(M(\nu); \mathbb{Z}_2) \xrightarrow{\mu_{4,2}} H^{2n}(M(\nu); \mathbb{Z}_4) \xrightarrow{\rho_1} H^{2n}(M(\nu); \mathbb{Z}_2) \xrightarrow{\rho_2} \cdots \]

If $\tau$ is a generator of $H^{2n}(M(\nu); \mathbb{Z})$ and if $\tau_2$ and $\tau_4$ are the reductions of $\tau$ mod 2 and mod 4, then the homomorphisms of the lower sequence are given by $\mu_{4,2}(\tau_2) = 2\tau_4$ and $\rho_2(\tau_4) = \tau_2$. It is clear that $\chi(\nu) \equiv 2 \mod 4$ if and only if $f^*(\chi_4) = 2\tau_4$. Now both $\chi_4$ and $p(w_n)$ go to $w_n^2$ under $\rho_2$. However, $\chi_4 \neq p(w_n)$, since $\beta_4(p(w_n)) \neq 0$. Thus $\chi_4 = p(w_n) + \mu_{4,2}(Y)$ for some $Y$ in $H^{2n}(MO(n); \mathbb{Z})$. To determine $Y$ I observe that the defining equation $\chi_4 = p(w_n) + \mu_{4,2}(Y)$ implies that

\[ Sq^1(Y) = \rho_2 \circ \beta_4 \circ \mu_{4,2}(Y) = \rho_2 \circ \beta_4(p(w_n)) \]

\[ = w_n \cdot Sq^1 w_n + Sq^2 w_n = w_n \cdot w_{n-1} \cdot w_1. \]

Direct computation shows that $Y$ is either $w_n \cdot w_{n-1} \cdot w_1$ or $w_n \cdot w_{n-1} \cdot w_1 + w_2^2$.

Now $f^*(\chi_4) = f^*(p(w_n) + \mu_{4,2}(Y)) = p(f^*(w_n)) + \mu_{4,2}(f^*(Y))$. Since the top dimensional cell of $M(\nu)$ is spherical, cohomology operations which take their value in dimension $2n$ are trivial for $M(\nu)$. In particular, $p(f^*(w_n)) = 0$. Since $w_n = 0$ for all manifolds, $f^*(w_n^2) = 0$, so...
\[ \ast(Y) = \ast(w_n \cdot w_{n-1} \cdot w_1). \]

The hypothesis that \( w_{n-1} \cdot w_1 \neq 0 \) implies that 

\[ \ast(w_n \cdot w_{n-1} \cdot w_1) = \tau_2. \]

Thus \( \ast(\chi_4) = 0 + \mu_4,2(\tau_2) = 2\tau_4 \), and \( \chi(\nu) \equiv 2 \mod 4. \)

Essentially the same procedure gives a converse:

**Theorem 3'**. If \( M^n \) is imbedded in \( \mathbb{R}^{2n} \) and if \( \chi(\nu) \equiv 2 \mod 4 \), then 

\( w_{n-1} \cdot w_1 \neq 0 \) and \( n = 2^k \).

**Proof.** I first note that the equation \( \chi_4 = p(w_n) + \mu_4,2(Y) \) holds for all \( n \). By hypothesis, \( \chi(\nu) \equiv 2 \mod 4 \), so 

\[ \ast(\chi_4) = 2\tau_4, \]

and 

\[ \ast(p(w_n)) = \mu_4,2(Y) = \mu_4,2(\ast(Y)) = 2\tau_4. \]

Thus 

\[ \ast(Y) = \ast(w_n \cdot w_{n-1} \cdot w_1) = \tau_2, \]

and 

\( w_{n-1} \cdot w_1 \neq 0 \). Massey has shown \([7]\) that 

\( w_{n-1} \cdot w_1 \neq 0 \) implies \( n \) is a power of 2. This completes the proof.

W. S. Massey has recently given a similar proof of this theorem \([5]\). A related but rather more involved proof is also to be found in \([6]\).

**References**


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