PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, No. 2, July 1971

NONSTABLE HOMOTOPY GROUPS OF THOM COMPLEXES

P. J. LEDDEN

ABSTRACT. The first nonstable homotopy group of $MO(n)$, $\pi_{2n}(MO(n))$, is computed for all n, together with the corresponding Postnikov invariant. The computations give a new proof of a theorem of Mahowald on the normal bundle of an imbedding.

1. In 1954 R. Thom introduced the complexes $MO(n)$ in his study of the global properties of differentiable manifolds: e.g., the realization of homology classes by submanifolds and cobordism, [1]. One of his central results is the computation of $\pi_{n+h}(MO(n))$ for $h<n$. Let $d(h)$ be the number of partitions, $\{a_1, \cdots, a_r\}$, of the integer h with no a_i of the form 2^s-1. (Thom calls these *nondyadic* partitions $d(1)=0$, $d(2)=1$, $d(3)=0$, $d(4)=2$, etc.) Then $\pi_{n+h}(MO(n))$ is the direct sum of $d(h)$ copies of Z_2, which I abbreviate: $d(h)\cdot Z_2$. Using the same methods as [1], I shall prove:

THEOREM 1. *For n an odd integer, $n\neq 2^r-1$, $\pi_{2n}(MO(n))=(d(n)+1)\cdot \boldsymbol{Z}_2$; for $n=2^r-1$, $\pi_{2n}(MO(n))=d(n)\cdot \boldsymbol{Z}_2$.*

THEOREM 2. *For n an even integer, $n\neq 2^r$, $\pi_{2n}(MO(n))=Z\oplus d(n)\cdot \boldsymbol{Z}_2$; for $n=2^r$, $\pi_{2n}(MO(n))=Z\oplus (d(n)-1)\cdot \boldsymbol{Z}_2$.*

The proofs contain somewhat more than the theorems state: the relevant Postnikov invariants are also obtained. The following section is concerned with the basic construction, information about $MO(n)$, and some technical facts about the Pontrjagin square. Theorem 1 is proven in §3 and Theorem 2 in §4. The last paragraph contains an application to manifolds.

2. I recall that $MO(n)$ is obtained from the universal disk bundle over $BO(n)$ by collapsing the entire boundary (i.e., the corresponding sphere bundle) to a point. $H^*(MO(n); \boldsymbol{Z}_2)$ can be identified, via the Thom isomorphism, φ^*, with the ideal generated by w_n in $H^*(BO(n); \boldsymbol{Z}_2)$ $=\boldsymbol{Z}_2[w_1, \cdots, w_n]$. For n an odd integer and p an odd prime, $\tilde{H}^*(MO(n); \boldsymbol{Z}_p)$ is trivial; for n even, $H^{2n+4j}(MO(n); \boldsymbol{Z}_p)=\boldsymbol{Z}_p$. Thom's computation of the stable homotopy of $MO(n)$ proceeds as

Received by the editors August 8, 1969 and, in revised form, September 25, 1970.
AMS 1969 *subject classifications*. Primary 5545, 5720.
Key words and phrases. Thom spaces, nonstable homotopy groups, orthogonal realization, normal bundle, imbedding theorems.

follows: (cf. pp. 36–42 of [1]). For each nondyadic partition $\omega_h = \{a_1, \cdots, a_r\}$ of h a distinguished element X_{ω_h} in $H^{n+h}(MO(n); \boldsymbol{Z}_2)$ is defined; each X_{ω_h} has associated with it a function $f_{\omega_h}: MO(n) \to K(\boldsymbol{Z}_2, n+h)$. The map

$$F_{2n-1} = \prod_{h<n} f_{\omega_h}: MO(n) \to \prod_{h<n} (K(\boldsymbol{Z}_2, n + h))^{d(h)} = K(2n - 1)$$

induces, for all coefficient fields, Z_p, cohomology isomorphisms in dimensions less than $2n$ and an injective homomorphism in dimension $2n$. Since the spaces $K(\boldsymbol{Z}_2, n+h)$ have no Z_p cohomology for p an odd prime, the only nontrivial case is $p=2$. The proof for Z_2 coefficients is based upon the following:

LEMMA 1. *Let $I=(i_1, \cdots, i_r)$ be an admissible sequence (i.e., $i_k \geqq 2i_{k+1}$ and $2i_1 - \sum_{k=1}^r i_k < n$); then the elements $Sq^{i_1} \circ \cdots \circ Sq^{i_r} X_{\omega_h} = Sq^I X_{\omega_h}$ with $i_1 \leqq n$ are linearly independent in $H^*(MO(n); \boldsymbol{Z}_2)$.*

Subject to the restriction that $\sum_{k=1}^r i_k < n$ this is proven by Thom (pp. 38–42 of [1]); the proof, in fact, carries through without significant change in the more general case, $i_1 \leqq n$. Now, a theorem of Serre says that the elements $Sq^I \epsilon$, where I is admissible and ϵ is the fundamental class in dimension k, form a basis for the algebra $H^*(\boldsymbol{Z}_2, k; \boldsymbol{Z}_2)$, [2]. Thus $F_{2n-1}^*: H^{n+h}(K(2n-1); \boldsymbol{Z}_2) \to H^{n+h}(MO(n); \boldsymbol{Z}_2)$ is injective for $h \leqq n$ and it is not hard to show that for $h<n$ it is also surjective. This means that F_{2n-1} gives isomorphisms in the same dimensions for all fields, $\boldsymbol{Z}_p$, and by the standard J. H. C. Whitehead-Hurewicz argument, $\pi_{n+h}(MO(n)) = \pi_{n+h}(K(2n-1)) = d(h) \cdot \boldsymbol{Z}_2$. Since $\pi_i(K(2n-1)) = 0$ for $i \geqq 2n$, $K(2n-1)$ is the $(2n-1)$st stage of the Postnikov tower for $MO(n)$.

It is important to note that the elements X_{ω_h} are not, in general, monomials in the ideal $\langle w_n \rangle$ of $H^*(BO(n); \boldsymbol{Z}_2)$. For $k \geqq 1$, however, $\{2^k\}$ is a nondyadic partition of $h=2^k$ and the corresponding X_{ω_h} is $w_n \cdot w_1^{2^k}$. (This is clear when one refers to Thom's definition of the X_{ω_h}.) The classes $w_n \cdot w_1^{2^k}$ are of some importance in the next two sections; to simplify the notation I shall denote $X_{\{2^k\}}$ by X_k, the corresponding map of $MO(n)$ to $K(\boldsymbol{Z}_2, n+2^k)$ by f_k, and the generator of the corresponding $H^{n+2^k}(Z_2, n+2^k; \boldsymbol{Z}_2)$ by ϵ_k. Thus for $k>0$, $f_k^*(\epsilon_k) = X_k = w_n \cdot w_1^{2^k}$. It is also convenient to define X_0 to be w_n, the corresponding map of $MO(n)$ to $K(\boldsymbol{Z}_2, n)$ to be f_0, and the generator of $H^n(Z_2, n; \boldsymbol{Z}_2)$ to be ϵ_0.

In §§4 and 5, I shall require some of the basic properties of the Pontrjagin square, $\mathfrak{p}$ [3], [4]. This is a cohomology operation of type $(n, \boldsymbol{Z}_2; 2n, \boldsymbol{Z}_4)$ and it is defined as follows. Let ζ be a cochain representing the class z in $H^n(X;\boldsymbol{Z}_2)$; $\delta\zeta=2\eta$ for some $(n+1)$ cochain η; $\mathfrak{p}(z)$ is the class defined by the mod 4 cocycle $\zeta\cup\zeta+\zeta\cup_1\delta\zeta$. Let

$$(1)\qquad \cdots \xrightarrow{Sq^1} H^i(X;\boldsymbol{Z}_2) \xrightarrow{\mu_{4,2}} H^i(X;\boldsymbol{Z}_4) \xrightarrow{\rho_2} H^i(X;\boldsymbol{Z}_2) \to \cdots$$

be the coefficient sequence derived from $0\to\boldsymbol{Z}_2\to\boldsymbol{Z}_4\to\boldsymbol{Z}_2\to 0$. It is not hard to show:

$$(1)\qquad \mu_{4,2}(z^2) = 2\mathfrak{p}(z),$$

$$(2)\qquad (\mathfrak{p}(z))_2 = z^2,$$

$$(3)\qquad \begin{aligned} (\beta_4(\mathfrak{p}(z)))_2 &= zSq^1z + Sq^nSq^1z \quad (n \text{ even}) \\ &= Sq^nSq^1z \qquad\qquad\quad (n \text{ odd}). \end{aligned}$$

Here β_4 is the Bockstein operator associated with the sequence $0\to\boldsymbol{Z}\to\boldsymbol{Z}\to\boldsymbol{Z}_4\to 0$ and $(\ )_2$ indicates reduction mod two. If n is even, $H^{2k+1}(\boldsymbol{Z}_2, k;\boldsymbol{Z})$ contains a $\boldsymbol{Z}_4$ summand which is generated by $\beta_4(\mathfrak{p}(\epsilon))$. (Proof: $(\beta_4(\mathfrak{p}(\epsilon)))_2=\epsilon\cdot Sq^1t+Sq^nSq^1\epsilon$; this last term is not of the form Sq^1x for some x in $H^{2k}(Z_2, k;\boldsymbol{Z}_2)$, so $\beta_4(\mathfrak{p}(\epsilon))$ is not of order two. It is of order $\leqq$ four because it is in the image of β_4.)

3. In this section n is an *odd* integer, $2^r<n<2^{r+1}$. The $2n$th stage of the Postnikov tower for $MO(n)$ is a fibration $p_{2n}:K_{2n}\to K_{2n-1}$ with fibre $K(\pi_{2n}(MO(n)), 2n)$ and a map $F_{2n}:MO(n)\to K_{2n}$ which induces homotopy isomorphisms in dimensions $\leqq 2n$. Moreover, the following diagram commutes:

$$\begin{tikzcd} & K_{2n} \arrow[dd, "P_{2n}"] \\ MO(n) \arrow[ur, "F_{2n}"] \arrow[dr, "F_{2n-1}"'] & \\ & K_{2n-1} \end{tikzcd}$$

As I noted above, to require that F_{2n} induce homotopy isomorphisms in the first $2n$ dimensions is equivalent to requiring that, for all coefficient fields $\boldsymbol{Z}_p$, F_{2n}^* be a cohomology isomorphism in dimensions $\leqq 2n$ and a monomorphism in dimension $2n+1$. The first step in the construction of K_{2n} is to obtain the cohomology isomorphism in dimension $2n$. Let $\overline{K}=K_{2n-1}\times(K(\boldsymbol{Z}_2, 2n))^{d(n)}$ and let $\overline{F}:MO(n)\to\overline{K}$ be the product of F_{2n-1} and the functions f_{ω_n} associated with the distinguished classes X_{ω_n}. (There are $d(n)$ of these classes and maps.) Both $MO(n)$ and $\overline{K}$ have trivial $\boldsymbol{Z}_p$ cohomology for p an odd prime (since n is odd) and Thom has shown than $\bar{F}^*$ is, in fact, an isomorphism in Z_2 cohomology in dimension $2n$. It remains now to determine the kernel of $\bar{F}^*$ in dimension $2n+1$ and to eliminate it by taking the appropriate fibration over $\bar{K}$. Z_2 coefficients are to be understood in the following

LEMMA 2. *If* $n=2^{r+1}-1$, $\bar{F}^*$ *is a monomorphism in dimension* $2n+1$. *If* $n\neq 2^{r+1}-1$, Ker $\bar{F}^*=Z_2$ *in dimension* $2n+1$; *the nonzero element is* $\theta=\epsilon_0\cdot Sq^1\epsilon_0+Sq^nSq^1\epsilon_0+Sq^{n-1}\epsilon_1+Sq^{n-3}\epsilon_2+\cdots+Sq^{n-2^r+1}\epsilon_r$.

PROOF OF THE LEMMA. I first note that $H^{2n+1}(\bar{K};Z_2)$ is the sum of the cohomology of the factors, since $\bar{K}=K(Z_2,n)\times K(Z_2,n+2)\times\cdots\times(K(Z_2,2n))^{d(n)}$, and the cross product terms of the Künneth formula (e.g. $\epsilon_0\cdot\epsilon_1$) do not appear until dimension $2n+2$. Thus the elements of $H^{2n+1}(\bar{K};Z_2)$ are of two kinds: those of the form $\epsilon_0\cdot Sq^1\epsilon_0+\sum Sq^I\epsilon_{\omega_k}$ and those of the form $\sum Sq^I\epsilon_{\omega_k}$. Ker $\bar{F}^*$ cannot contain elements of the second kind, for $\bar{F}^*(Sq^I\epsilon_{\omega_k})=Sq^IX_{\omega_k}$ and these terms are linearly independent. Moreover, if there is an element of the first kind in Ker $\bar{F}^*$ it is unique. (Proof: Let θ_1 and θ_2 be two such elements. Then $\theta_1+\theta_2$ is also in the kernel and it is of the second kind, since the $\epsilon_0\cdot Sq^1\epsilon_0$ terms cancel. Hence $\theta_1+\theta_2=0$.)

Now consider $\theta=\epsilon_0\cdot Sq^1\epsilon_0+Sq^nSq^1\epsilon_0+Sq^{n-1}\epsilon_1+\cdots+Sq^{n-2^r+1}\epsilon_r$. If $n\neq 2^{r+1}-1$,

$$\bar{F}^*(\theta)=w_n^2\cdot w_1+w_n^2\cdot w_1+w_n\cdot w_{n-1}\cdot w_1^2+w_n\cdot w_{n-1}\cdot w_1^2$$
$$+\cdots+w_n\cdot w_{n-2^r+1}\cdot w_1^{2^r}+w_n\cdot w_{n-2^r+1}\cdot w_1^{2^r}=0,$$

and the lemma is proven. If $n=2^{r+1}-1$, $\bar{F}^*(\theta)=w_n\cdot w_1^{2^{r+1}}=X_{r+1}$. Suppose that θ_1 is in Ker $\bar{F}^*$. Then $\theta_1+\theta$ is an element of the second kind, i.e., $\theta_1+\theta=\sum Sq^I\epsilon_{\omega_k}$, so $X_{r+1}=\bar{F}^*(\theta_1+\theta)=\bar{F}^*(\sum Sq^I\epsilon_{\omega_k})=\sum Sq^IX_{\omega_k}$. But this would be a relation among the elements which Lemma 1 states are independent. Hence in this case F^* is a monomorphism.

PROOF OF THEOREM 1. Let $n=2^{r+1}-1$; the space $\bar{K}$ and the map $\bar{F}$ satisfy the requirements for the $2n$th stage of the Postnikov tower. Since $\bar{K}=K_{2n-1}\times(K(Z_2,2n))^{d(n)}$, $\pi_{2n}(MO(n))=d(n)\cdot Z_2$ and the k-invariant is zero.

Now suppose $n\neq 2^{r+1}-1$. The cohomology class θ corresponds to a map $f_\theta:\bar{K}\to K(Z_2,2n+1)$; let K_{2n} be the fibre space over $\bar{K}$ induced by f_θ from the standard path fibration over $K(Z_2,2n+1)$. The fibre space K_{2n} is nontrivial only over the spaces $K(Z_2,n+2^k)$ which correspond to the classes ϵ_k, $k\leqq r$. Since $2^r<n$, these spaces $K(Z_2,n+2^k)$ have trivial homotopy in dimension $2n$. Thus the homo-

topy sequence of the fibration $p_{2n}: K_{2n} \to \bar{K}$ gives $\pi_{2n}(K_{2n})$ as the trivial extension of $\pi_{2n}(K) = d(n) \cdot \mathbf{Z}_2$ by $\pi_{2n}(K(\mathbf{Z}_2, 2n)) = \mathbf{Z}_2$. Therefore $\pi_{2n}(K_{2n}) = (d(n)+1) \cdot \mathbf{Z}_2$. Since $\bar{F}^*(\theta) = 0$, the map $\bar{F}$ can be lifted to a map $F_{2n}: MO(n) \to K_{2n}$. F_{2n} induces homotopy isomorphisms in dimensions $\leqq 2n$, so $\pi_{2n}(MO(n)) = (d(n)+1) \cdot \mathbf{Z}_2$. The k-invariant is the characteristic class of the fibration $p_{2n}: K_{2n} \to K_{2n-1}$, which is just the class θ. This completes the proof of Theorem 1.

4. In this section n is an *even* integer, $2^r \leqq n < 2^{r+1}$. The procedure is the same as in §3, but the situation is made slightly more complicated by the presence of an infinite cyclic summand in $H^{2n}(MO(n); \mathbf{Z})$. (The presence of this summand is indicated by the fact that $H^{2n}(MO(n); \mathbf{Z}_p) = Z_p$ for all odd primes.) The first step is to consider $\bar{F}: MO(n) \to \bar{K} = K_{2n-1} \times (K(\mathbf{Z}_2, 2n))^{d(n)}$. $\bar{F}^*$ is an isomorphism in $\mathbf{Z}_2$ *cohomology* in dimension $2n$ and the analogue of Lemma 2 also is true:

LEMMA 2′. *In $\mathbf{Z}_2$ cohomology in dimension* $2n+1$, Ker $\bar{F}^* = \mathbf{Z}_2$; *the generator is* $\theta = \epsilon_0 \cdot Sq^1\epsilon_0 + Sq^n Sq^1 \epsilon_0 + Sq^{n-1}\epsilon_1 + \cdots + Sq^{n-2^r+1}\epsilon_r$.

The proof is precisely the same as that of Lemma 2.

Since n is even, $Sq^{n-1}\epsilon_1 + Sq^{n-3}\epsilon_2 + \cdots + Sq^{n-2^r+1}\epsilon_r$ is in the image of Sq^1, so it is the reduction of some integral class, ζ, of order two. Now let $\mu = \beta_4(\mathfrak{p}(\epsilon_0)) + \zeta$; μ is an integral class of order four (since $\beta_4(\mathfrak{p}(\epsilon_0))$ is) and $\mu_2 = \epsilon_0 \cdot Sq^1\epsilon_0 + Sq^n Sq^1 \epsilon + \zeta_2 = \theta$. The class μ determines a mapping $f_\mu: \bar{K} \to K(\mathbf{Z}, 2n+1)$; let K_{2n} be the fibre space over K induced by f_μ from the path fibration over $K(\mathbf{Z}, 2n+1)$. $(\bar{F}^*(\mu))_2 = \bar{F}^*(\mu_2) = \bar{F}^*(\theta) = 0$. Since the integral cohomology of $MO(n)$ in dimension $2n+1$ has only elements of order two, this means that $\bar{F}^*(\mu) = 0$. Thus $\bar{F}$ can be lifted to a map F_{2n} of $MO(n)$ to K_{2n}. It is clear that F_{2n} induces the appropriate isomorphisms and monomorphisms in cohomology and that K_{2n} is the $2n$th stage of the Postnikov system. It remains to compute $\pi_{2n}(K_{2n})$. The fibration $p_{2n}: K_{2n} \to \bar{K}$ is nontrivial over the factors $K(\mathbf{Z}_2, n+2^k)$ corresponding to the classes ϵ_r, $k \leqq r$. Thus if $n \neq 2^r$ the homotopy sequence of the fibration $p_{2n}: K_{2n} \to \bar{K}$ has the trivial extension of $\pi_{2n}(\bar{K})$ by $\pi_{2n}(K(\mathbf{Z}, 2n))$ and so $\pi_{2n}(K_{2n}) = \mathbf{Z} \oplus d(n)\mathbf{Z}_2$. (This is equivalent to saying that the fibre of $p_{2n}: K_{2n} \to K_{2n-1}$ is $K(\mathbf{Z}, 2n) \times (K(\mathbf{Z}_2, 2n))^{d(n)}$.) If $n = 2^r$, the fibration over the factor $K(\mathbf{Z}_2, 2n)$ which corresponds to the class ϵ_r is nontrivial, since $Sq^1\epsilon_r$ is a summand of θ. Thus the homotopy sequence of the fibration $K_{2n} \to \bar{K}$ contains the nontrivial extension $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$, and $\pi_{2n}(K_{2n}) = \mathbf{Z} \oplus (d(n)-1)\mathbf{Z}_2$. This is equivalent to the fact that the fibre of $p_{2n}: K_{2n} \to K_{2n-1}$ is $K(\mathbf{Z}, 2n) \times (K(\mathbf{Z}_2, 2n))^{d(n)-1}$.

If $n \neq 2^r$, the Postnikov invariant in dimension $2n+1$ is μ; if $n = 2^r$, the invariant, which is the characteristic class of the fibration $p_{2n}: K_{2n} \to K_{2n-1}$, is $2\mu = 2\beta_4 \mathfrak{p}(\epsilon_0)$. This completes the proof of Theorem 2. I remark that this computation was made by Thom in the case $n=2$.

5. The preceding computations give a short proof of a theorem of Mahowald [**4**].

THEOREM 3. *Let M^n be a manifold of dimension $n=2^k$ with $\bar{w}_{n-1} \cdot \bar{w}_1 \neq 0$. For any imbedding of M^n in $\mathbf{R}^{2n}$ with normal bundle ν, the Euler characteristic $\chi(\nu)$ is congruent to two modulo four.*

PROOF. Let $f: M(\nu) \to MO(n)$ be the map of Thom complexes induced from the classifying map for ν, and let χ and $\chi(\nu)$ be the integral Euler classes of $MO(n)$ and $M(\nu)$. (That is to say, χ and $\chi(\nu)$ are the images under the Thom isomorphism of the Euler classes of the respective bundles.) Consider the following portions of the coefficient sequences of $MO(n)$ and $M(\nu)$ derived from the sequence $0 \to \mathbf{Z}_2 \to \mathbf{Z}_4 \to \mathbf{Z}_2 \to 0$.

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{Sq^1} & H^{2n}(MO(n); Z_2) & \xrightarrow{\mu_{4,2}} & H^{2n}(MO(n); Z_4) & \xrightarrow{\rho_2} & H^{2n}(MO(n); Z_2) & \xrightarrow{Sq^1} \cdots \\
& \downarrow{\scriptstyle \hat{f}^*} & & \downarrow{\scriptstyle \hat{f}^*} & & \downarrow{\scriptstyle \hat{f}^*} & \\
\cdots \xrightarrow{0} & H^{2n}(M(\nu); Z_2) & \xrightarrow{\mu_{4,2}} & H^{2n}(M(\nu); Z_4) & \xrightarrow{\rho_2} & H^{2n}(M(\nu); Z_2) & \longrightarrow 0 \cdots
\end{array}$$

If τ is a generator of $H^{2n}(M(\nu); Z)$ and if τ_2 and τ_4 are the reductions of τ mod 2 and mod 4, then the homomorphisms of the lower sequence are given by $\mu_{4,2}(\tau_2) = 2\tau_4$ and $\rho_2(\tau_4) = \tau_2$. It is clear that $\chi(\nu) \equiv 2 \bmod 4$ if and only if $\hat{f}^*(\chi_4) = 2\tau_4$. Now both χ_4 and $\mathfrak{p}(w_n)$ go to w_n^2 under ρ_2. However, $\chi_4 \neq \mathfrak{p}(w_n)$, since $\beta_4(\chi_4) = 0$ and $\beta_4(\mathfrak{p}(w_n)) \neq 0$. Thus $\chi_4 = \mathfrak{p}(w_n) + \mu_{4,2}(Y)$ for some Y in $H^{2n}(MO(n); Z_2)$. To determine Y I observe that the defining equation $\chi_4 = \mathfrak{p}(w_n) + \mu_{4,2}(Y)$ implies that

$$\begin{aligned}
Sq^1(Y) &= \rho_2 \circ \beta_4 \circ \mu_{4,2}(Y) = \rho_2 \circ \beta_4(\mathfrak{p}(w_n)) \\
&= w_n \cdot Sq^1 w_n + Sq^n Sq^1 w_n = w_n \cdot w_{n-1} \cdot w_1^2.
\end{aligned}$$

Direct computation shows that Y is either $w_n \cdot w_{n-1} \cdot w_1$ or $w_n \cdot w_{n-1} \cdot w_1 + w_n^2$.

Now $\hat{f}^*(\chi_4) = \hat{f}^*(\mathfrak{p}(w_n) + \mu_{4,2}(Y)) = \mathfrak{p}(\hat{f}^*(w_n)) + \mu_{4,2}(\hat{f}^*(Y))$. Since the top dimensional cell of $M(\nu)$ is spherical, cohomology operations which take their value in dimension $2n$ are trivial for $M(\nu)$. In particular, $\mathfrak{p}(\hat{f}^*(w_n)) = 0$. Since $\bar{w}_n = 0$ for all manifolds, $\hat{f}^*(w_n^2) = 0$, so

$\hat{f}^*(Y) = \hat{f}^*(w_n \cdot w_{n-1} \cdot w_1)$. The hypothesis that $\bar{w}_{n-1} \cdot \bar{w}_1 \neq 0$ implies that $\hat{f}^*(w_n \cdot w_{n-1} \cdot w_1) = \tau_2$. Thus $\hat{f}^*(\chi_4) = 0 + \mu_{4,2}(\tau_2) = 2\tau_4$, and $\chi(\nu) \equiv 2 \bmod 4$.

Essentially the same procedure gives a converse:

THEOREM 3′. *If M^n is imbedded in R^{2n} and if $\chi(\nu) \equiv 2 \bmod 4$, then $\bar{w}_{n-1} \cdot w_1 \neq 0$ and $n = 2^k$.*

PROOF. I first note that the equation $\chi_4 = \mathfrak{p}(w_n) + \mu_{4,2}(Y)$ holds for all n. By hypothesis, $\chi(\nu) \equiv 2 \bmod 4$, so $\hat{f}^*(\chi_4) = 2\tau_4$, and $\hat{f}^*(\mathfrak{p}(w_n)) = \mu_{4,2}(Y) = \mu_{4,2}(\hat{f}^*(Y)) = 2\tau_4$. Thus $\hat{f}^*(Y) = \hat{f}^*(w_n \cdot w_{n-1} \cdot w_1) = \tau_2$, and $\bar{w}_{n-1} \cdot \bar{w}_1 \neq 0$. Massey has shown [7] that $\bar{w}_{n-1} \cdot w_1 \neq 0$ implies n is a power of 2. This completes the proof.

W. S. Massey has recently given a similar proof of this theorem [5]. A related but rather more involved proof is also to be found in [6].

REFERENCES

1. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86. MR **15**, 890.

2. J. P. Serre, *Cohomologie modulo 2 des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. **27** (1953), 198–232. MR **15**, 643.

3. J. H. C. Whitehead, *On simply connected, 4-dimensional polyhedra*, Comment. Math. Helv. **22** (1949), 48–92. MR **10**, 559.

4. M. Mahowald, *On the normal bundle of a manifold*, Pacific J. Math. **14** (1964), 1335–1341. MR **31** #757.

5. W. S. Massey, *Pontrjagin squares in the Thom space of bundle*, Pacific J. Math. (to appear).

6. P. J. Ledden, *Thom spaces, projective spaces, and a theorem of Whitney*, Thesis, Stanford University, Stanford, Calif., 1965.

7. W. S. Massey, *On the Stiefel-Whitney classes of a manifold*, Amer. J. Math. **82** (1960), 92–102. MR **22** #1918.

8. ———, *Proof of a conjecture of Whitney*, Pacific J. Math. (to appear)

UNIVERSITY OF CALIFORNIA AT SAN DIEGO, LA JOLLA, CALIFORNIA 92037