A CHARACTERIZATION OF COMPACT GROUPS

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Abstract. A locally compact group is compact if and only if it is of type I and has discrete spectrum (equivalently: its C*-algebra is dual).

For separable groups, the present result is Theorem 3.4 of [1]. (In this case, the type I assumption is redundant; whether this is so in general seems to be an unsolved case of a problem of Naimark.) Upon dropping the hypothesis of separability, one would hope to avoid introducing it during the proof. We have, however, been forced to refer to [1], and indeed we do very little else.

Theorem. Suppose that $G$ is a type I locally compact group with discrete spectrum. Then $G$ is compact.

Proof. Let us show first that open subgroups of $G$ and quotient groups of $G$ also satisfy the hypotheses. The $C^*$-algebras of such groups are, respectively, sub-$C^*$-algebras and quotient $C^*$-algebras of the $C^*$-algebra of $G$. By 10.10.6 of [2], and the fact [3, Main Theorem] that a type I $C^*$-algebra is postliminary (so that if it is simple it is elementary), the $C^*$-algebra of $G$ is dual. A sub-$C^*$-algebra and a quotient $C^*$-algebra of a dual $C^*$-algebra are dual. A dual $C^*$-algebra is type I and has discrete spectrum.

To prove the Theorem, inspection of §III of [1] shows that it is enough to prove that $G$ has a compact open subgroup. As in the proof of Theorem 2.7 of [1], let us start with the fact that $G$, a locally compact group, has an open subgroup $H$ with a compact normal subgroup $K$ such that $H/K$ is Lie. The connected component of the identity in $H/K$ is open, and is separable; hence by the preceding paragraph and by Corollary 2.6 of [1] it is compact. Its inverse image in $H$ is open and compact.

References

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