PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, August 1971

A BRIEF PROOF OF CAUCHY'S INTEGRAL THEOREM

JOHN D. DIXON[1]

ABSTRACT. A short proof of Cauchy's theorem for circuits homologous to 0 is presented. The proof uses elementary local properties of analytic functions but no additional geometric or topological arguments.

The object of this note is to present a very short and transparent proof of Cauchy's theorem for circuits homologous to 0. The proof is based on simple 'local' properties of analytic functions that can be derived from Cauchy's theorem for analytic functions on a disc, and it may be compared with the treatment in Ahlfors [1, pp. 137–145]. It is apparent from this proof that this version of Cauchy's theorem is not only much more natural than the homotopic version which appears in several recent textbooks; it is also much easier to prove (contra Dieudonné [2, p. 192]). It is reasonable to argue that the concept of homotopy in connection with Cauchy's theorem is as extraneous as the notion of Jordan curve.

We recall that if γ is a circuit (= "continuous, piecewise smooth, closed curve"), and $w \in \mathbf{C}$ does not lie on γ, then the *index* of w with respect to γ is $\operatorname{Ind}(\gamma, w) = (2\pi i)^{-1}\int_\gamma (z-w)^{-1}dz$. It is easily proved that $E = \{w \in \mathbf{C} \mid \operatorname{Ind}(\gamma, w) = 0\}$ contains a neighbourhood of ∞ and is open (see [1, p. 116]). In the following proof we give full references to the 'local' properties used in order to emphasize the elementary nature of the proof.

CAUCHY'S THEOREM. *Let D be an open subset of $\mathbf{C}$ and let γ be a circuit in D. Suppose that γ is homologous to 0 in D, i.e. each $w \notin D$ lies in the set E defined above. Then, for each f analytic on D:*

(i) $\int_\gamma f(z)dz = 0$;

(ii) $\operatorname{Ind}(\gamma, w)f(w) = (2\pi i)^{-1}\int_\gamma (z-w)^{-1}f(z)dz$ *for all $w \in D$ not lying on γ.*

PROOF. Consider $g: D \times D \to \mathbf{C}$ defined by $g(w, z) = (f(z) - f(w))/(z-w)$ for $z \neq w$ and $g(w, w) = f'(w)$. Then g is continuous, and for each fixed z, $w \mapsto g(w, z)$ is analytic [1, p. 124]. Define $h: \mathbf{C} \to \mathbf{C}$ by $h(w)$

Received by the editors January 18, 1971.

AMS 1970 *subject classifications*. Primary 30X01, 30A86.

Key words and phrases. Cauchy's integral theorem, Cauchy's integral formula, residue theorem.

[1] This research was supported in part by the National Research Council of Canada (Grant No. A7171).

$=\int_\gamma g(w, z)dz$ on D and $h(w)=\int_\gamma (z-w)^{-1}f(z)dz$ on E. Note that $\mathbf{C}=D\cup E$ by hypothesis, and that these two expressions for $h(w)$ are equal on $D\cap E$ because $\operatorname{Ind}(\gamma, w)=0$ there.

Now h is differentiable on both D and E ([**1**, p. 123] or [**3**, p. 137]), and so h is an entire function. Since the image of γ is bounded, and E contains a neighbourhood of ∞, $h(w)\to 0$ as $w\to\infty$. This implies firstly that h is constant (Liouville's theorem), and secondly that h is 0. Thus $\int_\gamma g(w, z)dz=0$ for all $w\in D$ not lying on γ; and (ii) follows. Finally, let u be some fixed point of D not lying on γ. Then applying (ii) to the function $z\mapsto f(z)(z-u)$ in place of f, and evaluating at $w=u$, we obtain (i).

REMARK. The proof goes through word for word when γ is a cycle (see [**1**, p. 138]) rather than a circuit. Then, as in Ahlfors' treatment, the general form of the residue theorem follows immediately.

REFERENCES

1. L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, 2nd ed., McGraw-Hill, New York, 1966. MR **32** #5844.

2. J. Dieudonné, *Foundations of modern analysis*, Pure and Appl. Math., vol. 10, Academic Press, New York, 1960. MR **22** #11074.

3. R. Nevanlinna and V. Paatero, *Einführung in die Funktionentheorie*, Lehrbücher und Monographien aus dem Gebiete der exakten Wissenschaften, Mathematische Reihe, Band 30, Birkhäuser Verlag, Basel, 1965; English transl., Addison-Wesley, Reading, Mass., 1969. MR **34** #1491; MR **39** #415.

CARLETON UNIVERSITY, OTTAWA 1, ONTARIO, CANADA