PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 29, Number 3, August 1971

RATIONAL APPROXIMATION ON THE UNION OF SETS

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ABSTRACT. A counterexample is given to a conjecture of Val'skiĭ that if K is a compact plane set with interior U and the continuous function f on K satisfies $f|\overline{U}\in R(\overline{U})$ and $f|bK\in R(bK)$ then $f\in R(K)$. The conjecture is shown to be true when U is a disc.

Let K be a compact subset of the complex plane $\mathbf{C}$. We denote by $C(K)$ the algebra of all continuous complex-valued functions on K, by $R(K)$ the subalgebra of all uniform limits on K of rational functions with poles outside K, and by $A(K)$ the algebra of all functions in $C(K)$ which are analytic on the interior of K. In [**2**] Val'skiĭ conjectured the following: Suppose U is the interior of K, and $f\in C(K)$ satisfies $f|\overline{U}\in R(\overline{U})$ and $f|bK\in R(bK)$. ($\overline{U}$ or $\mathrm{cl}(U)$ denote the closure of U, bK the boundary of K.) Then $f\in R(K)$. He also mentioned the special case when U is the open unit disc Δ. In the present paper a counterexample is given to the general conjecture but it is shown to be true in the case of the disc.

THEOREM 1. *There exists a compact set K with interior U and $f\in C(K)$ such that $f|\overline{U}\in R(\overline{U})$, $f|bK\in R(bK)$, but $f\notin R(K)$.*

PROOF. Let Γ be an arc with endpoints -1 and 1 lying in Δ, with positive continuous analytic capacity [**1**, p. 203]. $\Delta\backslash\Gamma$ has two components U and V say. Let $\{\Delta_j\}_{j=1}^{\infty}$ be a sequence of disjoint open discs in V such that $V\backslash\bigcup_{j=1}^{\infty}\Delta_j$ is nowhere dense, and the sum of the radii of the discs is finite. Let $K=\overline{\Delta}\backslash\bigcup_{j=1}^{\infty}\Delta_j$. Then the interior of K is U and $bK=b\Delta\cup\Gamma\cup\mathrm{cl}(\bigcup_{j=1}^{\infty}b\Delta_j)$. Since $\alpha(\Gamma)>0$, there exists an f continuous on $\mathbf{C}$ and analytic outside Γ such that $f'(\infty)\neq 0$. Let μ be the measure on bK which coincides with dz on $b\Delta$, with $-dz$ on $\bigcup_{j=1}^{\infty}b\Delta_j$ and is zero on the rest of bK. Then $\int g\,d\mu=0$ for every rational function with poles outside K and hence $\int g\,d\mu=0$ for all $g\in R(K)$. Since $\int_{bK}f\,d\mu=\int_{b\Delta}f\,dz=2\pi i\cdot f'(\infty)\neq 0$, $f|K\notin R(K)$. Since $\overline{U}$ has connected complement, $R(\overline{U})=A(\overline{U})$ and so $f|\overline{U}\in R(\overline{U})$. Likewise $\mathbf{C}\backslash(\overline{V}\cup bU)$ has just two components, so $R(\overline{V}\cup bU)$

Received by the editors October 23, 1970.

AMS 1969 *subject classifications*. Primary 3070.

Key words and phrases. Compact plane set, rational approximation, T_φ operator, analytically negligible set.

[1] This work was partially supported by NSF Grant GP-19067.

[2] This work was partially supported by NSF Grant GP-11475.

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$=A(\overline{V}\cup bU)$ and hence $f|\overline{V}\cup bU\in R(\overline{V}\cup bU)$. Since $bK\subset\overline{V}\cup bU$, it follows that $f|bK\in R(bK)$.

Before proving the second theorem we make some remarks about the T_φ operator. Let φ be a C^1 function with compact support and f be a continuous function on $\mathbf{C}$. We define $T_\varphi f$ by

$$T_\varphi f(\zeta) = \varphi(\zeta)f(\zeta) + \frac{1}{\pi}\cdot\int \frac{f(z)}{z-\zeta}\cdot\frac{\partial\varphi}{\partial\bar{z}}\,dm(z), \qquad \zeta\in\mathbf{C},$$

where m denotes plane Lebesgue measure. Then $T_\varphi f$ is continuous where f is, analytic outside the support of φ and on any open set where f is analytic. See for example [1, p. 29].

THEOREM 2. *Let $Q\subseteq K$ be compact. Let $f\in C(K)$ such that each $z\in K\backslash Q$ has a neighbourhood V with $f|\operatorname{cl}(K\cap V)\in R(\operatorname{cl}(K\cap V))$. Let $\epsilon>0$. Then there exists g, continuous on $\mathbf{C}$ and analytic on a neighbourhood of $K\backslash Q$, with $|f-g|<\epsilon$ on K.*

PROOF. Extend f to be continuous on $\mathbf{C}$ with compact support. Let $M_n=\{z\in K;\ d(z, Q)\geqq 2^{-n}\}$, $n=1, 2, \cdots$. Then M_n is compact and $K\backslash Q=\bigcup_n M_n$. We construct by induction on n sequences $\{W_n\}$, $\{f_n\}$, where W_n is a compact neighbourhood of M_n, $W_n\subseteq W_{n+1}$, and f_n is continuous on $\mathbf{C}$ with compact support and analytic on a neighbourhood of W_n and moreover each $z\in K\backslash Q$ has a neighbourhood V with $f_n|\operatorname{cl}(K\cap V)\in R(\operatorname{cl}(K\cap V))$, $\|f_1-f\|<\epsilon/2$ and $\|f_{n+1}-f_n\|<\epsilon/2^{n+1}$ for all n.

Inductive step (*initial step similar*). Suppose f_n, W_n constructed. Let U be a neighbourhood of W_n on which f_n is analytic. Let $\Delta_1, \cdots, \Delta_r$ be a covering of M_{n+1} by open discs such that $f_n|\operatorname{cl}(K\cap\Delta_k)\in R(\operatorname{cl}(K\cap\Delta_k))$ for each k and such that any Δ_k which meets W_n must lie in U. Let $\varphi_1, \cdots, \varphi_r$ be C^1 functions, φ_k supported on a compact subset of Δ_k, with $\sum_{k=1}^r \varphi_k=1$ on a neighbourhood of M_{n+1}. Fix k; we can find a sequence $\{g_s\}$, g_s continuous on $\mathbf{C}$ and analytic on a neighbourhood of $\operatorname{cl}(K\cap\Delta_k)$, $g_s\to f_n$ uniformly. Then $T_{\varphi_k}g_s\to T_{\varphi_k}f_n$ uniformly; moreover $T_{\varphi_k}g_s$ is analytic in a neighbourhood of K. For each k we define F_k as follows: If Δ_k meets W_n, put $F_k=0$ (in this case $T_{\varphi_k}f_n=0$). If Δ_k does not meet W_n choose g_s such that

$$\|T_{\varphi_k}g_s - T_{\varphi_k}f_n\| < \epsilon/(2^{n+1}\cdot r)$$

and put $F_k=T_{\varphi_k}g_s$. Then F_k is analytic in a neighbourhood of $K\cup W_n$. Put

$$f_{n+1} = f_n - \sum_{k=1}^r T_{\varphi_k}f_n + \sum_{k=1}^r F_k.$$

Since $\sum_{k=1}^{r} \varphi_k = 1$ on a neighbourhood of M_{n+1}, $f_n - \sum_{k=1}^{r} T_{\varphi_k} f_n$ is analytic in a neighbourhood of M_{n+1}. Also it is analytic on a neighbourhood of W_n since f_n is. Hence f_{n+1} is analytic on a neighbourhood of $M_{n+1} \cup W_n$. Let W_{n+1} be a compact neighbourhood of $W_n \cup M_{n+1}$ such that f_{n+1} is analytic on a neighbourhood of W_{n+1}. That each $z \in K \backslash Q$ has a neighbourhood V with $f_{n+1} | \operatorname{cl}(K \cap V) \in R(\operatorname{cl}(K \cap V))$ follows from the fact that each $T_{\varphi_k} f_n$ and each F_k has this property. Finally

$$\|f_{n+1} - f_n\| \leqq \sum_{k=1}^{r} \|F_k - T_{\varphi_k} f_n\| < \epsilon/2^{n+1}.$$

This completes the induction.

We have $f_n \to g$ say, uniformly, where g is analytic on the interior of W_n for each n; hence on a neighbourhood of $\bigcup_n M_n = K \backslash Q$. Finally

$$\|f - g\| \leqq \|f - f_1\| + \sum_{n=1}^{\infty} \|f_{n+1} - f_n\| < \epsilon,$$

and the proof is complete.

A compact set E is *analytically negligible* if every continuous function on $\mathbf{C}$ which is analytic on an open set V can be approximated uniformly on $\mathbf{C}$ by functions continuous on $\mathbf{C}$ and analytic on $V \cup E$. Since a circle is analytically negligible [1, 8.12.3], the following result answers Val'skiĭ's question affirmatively when U is a disc:

COROLLARY. *Let K, L be a compact sets such that $bK \cap bL$ is analytically negligible. Let $f \in C(K \cup L)$ such that $f | K \in R(K)$ and $f | L \in R(L)$. Then $f \in R(K \cup L)$.*

PROOF. Theorem 2 applies with $K \cup L$ in place of K and $Q = bK \cap bL$. Given $\epsilon > 0$, we can find g continuous on $\mathbf{C}$ and analytic on a neighbourhood U of $K \cup L \backslash (bK \cap bL)$ with $|f - g| < \epsilon/2$ on K. Since $bK \cap bL$ is analytically negligible, we can find h, continuous on $\mathbf{C}$ and analytic on U and on a neighbourhood of $bK \cap bL$, $\|h - g\| < \epsilon/2$. Then h is analytic on a neighbourhood of $K \cup L$ and $|f - h| < \epsilon$ on K. Hence $f \in R(K \cup L)$.

Some open problems. The following questions are closely related to the contents of this paper.

(1) Suppose $f \in C(K)$ and that for each $z \in K$ where f does not vanish we can find a neighbourhood V of z such that $f | (\overline{V} \cap K) \in R(\overline{V} \cap K)$.

Must $f \in R(K)$?

Theorem 2 shows that the answer is yes provided the zero set of

f is analytically negligible. An affirmative answer would also settle the following question due to Björk (private communication).

(2) Suppose $f \in C(K)$ and $f^2 \in R(K)$. Must $f \in R(K)$?

The final question is a somewhat weaker version of (1) which we have nevertheless been unable to prove.

(3) Suppose $f \in C(K \cup L)$ satisfies $f|K \in R(K)$ and $f=0$ on L. Must $f \in R(K \cup L)$?

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