RATIONAL APPROXIMATION ON THE UNION OF SETS

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Abstract. A counterexample is given to a conjecture of Val'skiïf that if \(K\) is a compact plane set with interior \(U\) and the continuous function \(f\) on \(K\) satisfies \(f\mid U \in R(\bar{U})\) and \(f\mid bK \in R(bK)\) then \(f \in R(K)\). The conjecture is shown to be true when \(U\) is a disc.

Let \(K\) be a compact subset of the complex plane \(C\). We denote by \(C(K)\) the algebra of all continuous complex-valued functions on \(K\), by \(R(K)\) the subalgebra of all uniform limits on \(K\) of rational functions with poles outside \(K\), and by \(A(K)\) the algebra of all functions in \(C(K)\) which are analytic on the interior of \(K\). In \([2]\) Val'skiï conjectured the following: Suppose \(U\) is the interior of \(K\), and \(f \in C(K)\) satisfies \(f\mid U \in R(\bar{U})\) and \(f\mid bK \in R(bK)\). (\(\bar{U}\) or \(cl(U)\) denote the closure of \(U\), \(bK\) the boundary of \(K\).) Then \(f \in R(K)\). He also mentioned the special case when \(U\) is the open unit disc \(A\). In the present paper a counterexample is given to the general conjecture but it is shown to be true in the case of the disc.

Theorem 1. There exists a compact set \(K\) with interior \(U\) and \(f \in C(K)\) such that \(f\mid U \in R(\bar{U}), f\mid bK \in R(bK)\), but \(f \notin R(K)\).

Proof. Let \(\Gamma\) be an arc with endpoints \(-1\) and \(1\) lying in \(\Delta\), with positive continuous analytic capacity \([1, \text{p. 203}]\). \(\Delta \setminus \Gamma\) has two components \(U\) and \(V\) say. Let \(\{\Delta_j\}_{j=1}^\infty\) be a sequence of disjoint open discs in \(V\) such that \(V \setminus \bigcup_{j=1}^\infty \Delta_j\) is nowhere dense, and the sum of the radii of the discs is finite. Let \(K = \Delta \setminus \bigcup_{j=1}^\infty \Delta_j\). Then the interior of \(K\) is \(U\) and \(bK = \partial \Delta \cup \Gamma \cup \text{cl}(\bigcup_{j=1}^\infty \partial \Delta_j)\). Since \(\alpha(\Gamma) > 0\), there exists an \(f\) continuous on \(C\) and analytic outside \(\Gamma\) such that \(f'(\infty) \neq 0\). Let \(\mu\) be the measure on \(bK\) which coincides with \(ds\) on \(b\Delta\), with \(-ds\) on \(U \setminus \bigcup_{j=1}^\infty b\Delta_j\), and is zero on the rest of \(bK\). Then \(\int g d\mu = 0\) for every rational function with poles outside \(K\) and hence \(\int g d\mu = 0\) for all \(g \in R(K)\). Since \(\int_{bK} f d\mu = \int_{b\Delta} f dz = 2\pi i \cdot f'(\infty) \neq 0\), \(f \notin R(K)\). Since \(\bar{U}\) has connected complement, \(R(\bar{U}) = A(\bar{U})\) and so \(f\mid \bar{U} \in R(\bar{U})\).

Likewise \(C\setminus(\bar{V} \cup bU)\) has just two components, so \(R(\bar{V} \cup bU)\)

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Before proving the second theorem we make some remarks about the $T_\varphi$ operator. Let $\varphi$ be a $C^1$ function with compact support and $f$ be a continuous function on $C$. We define $T_\varphi f$ by

$$T_\varphi f(\zeta) = \varphi(f) f(\zeta) + \frac{1}{\pi} \int \frac{f(z) \, \partial \varphi}{z - \zeta} \, dm(z), \quad \zeta \in C,$$

where $m$ denotes plane Lebesgue measure. Then $T_\varphi f$ is continuous where $f$ is, analytic outside the support of $\varphi$ and on any open set where $f$ is analytic. See for example [1, p. 29].

**Theorem 2.** Let $Q \subseteq K$ be compact. Let $f \in C(K)$ such that each $z \in K \setminus Q$ has a neighbourhood $V$ with $f \big| (K \cap V) \in R(cl(K \cap V))$. Let $\epsilon > 0$. Then there exists $g$, continuous on $C$ and analytic on a neighbourhood of $K \setminus Q$, with $|f - g| < \epsilon$ on $K$.

**Proof.** Extend $f$ to be continuous on $C$ with compact support. Let $M_n = \{z \in K; d(z, Q) \geq 2^{-n}\}$, $n = 1, 2, \ldots$. Then $M_n$ is compact and $K \setminus Q = \bigcup_n M_n$. We construct by induction on $n$ sequences $\{W_n\}$, $\{f_n\}$, where $W_n$ is a compact neighbourhood of $M_n$, $W_n \subseteq W_{n+1}$, and $f_n$ is continuous on $C$ with compact support and analytic on a neighbourhood of $W_n$ and moreover each $z \in (K \setminus Q)$ has a neighbourhood $V$ with $f_n \big| (K \cap V) \in R(cl(K \cap V))$, $\|f_n - f\| < \epsilon/2$ and $\|f_{n+1} - f_n\| < \epsilon/2^{n+1}$ for all $n$.

**Inductive step (initial step similar).** Suppose $f_n$, $W_n$ constructed. Let $U$ be a neighbourhood of $W_n$ on which $f_n$ is analytic. Let $\Delta_1, \ldots, \Delta_r$ be a covering of $M_{n+1}$ by open discs such that $f_n \big| (K \setminus \Delta_k) \in R(cl(K \setminus \Delta_k))$ for each $k$ and such that any $\Delta_k$ which meets $W_n$ must lie in $U$. Let $\varphi_1, \ldots, \varphi_r$ be $C^1$ functions, $\varphi_k$ supported on a compact subset of $\Delta_k$, with $\sum_{k=1}^r \varphi_k = 1$ on a neighbourhood of $M_{n+1}$. Fix $k$; we can find a sequence $\{g_n\}$, $g_n$ continuous on $C$ and analytic on a neighbourhood of $cl(K \setminus \Delta_k)$, $g_n \to f_n$ uniformly. Then $T_\varphi g_n \to T_\varphi f_n$ uniformly; moreover $T_\varphi g_n$ is analytic in a neighbourhood of $K$. For each $k$ we define $F_k$ as follows: If $\Delta_k$ meets $W_n$, put $F_k = 0$ (in this case $T_\varphi f_n = 0$). If $\Delta_k$ does not meet $W_n$ choose $g_n$ such that

$$\|T_\varphi g_n - T_\varphi f_n\| < \epsilon/(2^{n+1}, r)$$

and put $F_k = T_\varphi g_n$. Then $F_k$ is analytic in a neighbourhood of $K \cup W_n$. Put

$$f_{n+1} = f_n - \sum_{k=1}^r T_\varphi f_n + \sum_{k=1}^r F_k.$$
Since \( \sum_{n=1}^{\infty} \varphi_n = 1 \) on a neighbourhood of \( M_{n+1} \), \( f_n - \sum_{n=1}^{\infty} T_{n,n+1} \) is analytic in a neighbourhood of \( M_{n+1} \). Also it is analytic on a neighbourhood of \( W_n \) since \( f_n \) is. Hence \( f_{n+1} \) is analytic on a neighbourhood of \( M_{n+1} \cup W_n \). Let \( W_{n+1} \) be a compact neighbourhood of \( W_n \cup M_{n+1} \) such that \( f_{n+1} \) is analytic on a neighbourhood of \( W_{n+1} \). That each \( z \in K \setminus Q \) has a neighbourhood \( V \) with \( f_{n+1} \mid (K \setminus V) \in R(\text{cl}(K \setminus V)) \) follows from the fact that each \( T_{n,n+1} \) and each \( F_k \) has this property. Finally

\[
\|f_{n+1} - f_n\| \leq \sum_{k=1}^{\infty} \|F_k - T_{n,n+1}\| < \frac{\epsilon}{2^{n+1}}.
\]

This completes the induction.

We have \( f_n \to g \) say, uniformly, where \( g \) is analytic on the interior of \( W_n \) for each \( n \); hence on a neighbourhood of \( \bigcup_n M_n = K \setminus Q \). Finally

\[
\|f - g\| \leq \|f - f_1\| + \sum_{n=1}^{\infty} \|f_{n+1} - f_n\| < \epsilon,
\]

and the proof is complete.

A compact set \( E \) is analytically negligible if every continuous function on \( C \) which is analytic on an open set \( V \) can be approximated uniformly on \( C \) by functions continuous on \( C \) and analytic on \( V \cup E \). Since a circle is analytically negligible \([1, 8.12.3]\), the following result answers Val'skin's question affirmatively when \( U \) is a disc:

**Corollary.** Let \( K, L \) be a compact sets such that \( bK \cap bL \) is analytically negligible. Let \( f \in C(K \cup L) \) such that \( f \mid K \in R(K) \) and \( f \mid L \in R(L) \). Then \( f \in R(K \cup L) \).

**Proof.** Theorem 2 applies with \( K \cup L \) in place of \( K \) and \( Q = bK \cap bL \). Given \( \epsilon > 0 \), we can find \( g \) continuous on \( C \) and analytic on a neighbourhood \( U \) of \( K \cup L \setminus (bK \cap bL) \) with \( \|f - g\| < \epsilon/2 \) on \( K \). Since \( bK \cap bL \) is analytically negligible, we can find \( h \), continuous on \( C \) and analytic on \( U \) and on a neighbourhood of \( bK \cap bL \), \( \|h - g\| < \epsilon/2 \). Then \( h \) is analytic on a neighbourhood of \( K \cup L \) and \( \|f - h\| < \epsilon \) on \( K \). Hence \( f \in R(K \cup L) \).

**Some open problems.** The following questions are closely related to the contents of this paper.

1. Suppose \( f \in C(K) \) and that for each \( z \in K \) where \( f \) does not vanish we can find a neighbourhood \( V \) of \( z \) such that \( f \mid (V \cap K) \in R(V \cap K) \). Must \( f \in R(K) \)?

Theorem 2 shows that the answer is yes provided the zero set of
$f$ is analytically negligible. An affirmative answer would also settle
the following question due to Björk (private communication).

(2) Suppose $f \in C(K)$ and $f^* \in R(K)$. Must $f \in R(K)$?

The final question is a somewhat weaker version of (1) which we
have nevertheless been unable to prove.

(3) Suppose $f \in C(K \cup L)$ satisfies $f|_K \in R(K)$ and $f = 0$ on $L$.
Must $f \in R(K \cup L)$?

REFERENCES


2. R. E. Val’skii, *On parts of algebras of analytic functions and measures orthogonal