A CLASS OF COMMUTATIVE BANACH ALGEBRAS
WITH UNIQUE COMPLETE NORM TOPOLOGY
AND CONTINUOUS DERIVATIONS

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Abstract. Let $A$ be a semisimple commutative complex algebra with identity and $\alpha(x)$ a monic polynomial over $A$. Two results are proved. If $B=A[x]/(\alpha(x))$ is a Banach algebra under some norm, then $B$ has a unique complete norm topology. Furthermore, $B$ has nontrivial derivations if and only if $B$ has a nontrivial radical.

Introduction. It is known that if $A$ is a commutative Banach algebra with identity and $\alpha(x)$ is a monic polynomial over $A$, then $B=A[x]/(\alpha(x))$ can be made into a Banach algebra with a norm extending the norm on $A$ [1]. In [2], D. T. Brown has shown that if $A$ is semisimple and regular (in the sense of Šilov), then $B$ has unique complete norm topology. The assumption of regularity can be dropped as we show in §1. In general, the extension $B$ is not semisimple; indeed, $B$ has a nontrivial radical if and only if the discriminant of $\alpha(x)$ is zero or a zero divisor in $A$ [5].

§2 is motivated by a recent result of R. J. Loy that states, in our context, that all derivations on $B$ are continuous [4]. However, he does not show the existence of derivations on (nonsemisimple) $B$ even though he gives their form. We consider the existence of nontrivial derivations in §2.

1. Uniqueness of complete norm topology on $A[x]/(\alpha(x))$. It is convenient to set first some of our notation and to make several comments. Lemma 1 of [2] states that if $A$ is a semisimple algebra (all algebras in this paper are assumed to be over the complex field $C$ and to possess an identity element) and if $B=A[x]/(\alpha(x))$ is a Banach algebra, then $A$ is a Banach algebra under some norm. Actually, it is proved that if $K(A)$ denotes the intersection of the kernels of nontrivial complex homomorphisms, then $A/K(A)$ is a Banach algebra. However, it is known that each maximal ideal $M$ in $A$ is contained in a maximal ideal $M_0$ in $B$ such that $M=M_0 \cap A$ (see p. 259 of [7]). Since $B$ is a Banach algebra, $M_0$ is the kernel of a com-
plex homomorphism; hence, so is $M$. Thus, $K(A) = (0)$ since $A$ is semisimple and Brown's Lemma 1 follows.

For a complex algebra $A$, let $\Phi_A$ denote the space of nontrivial complex homomorphisms on $A$, together with the Gelfand topology (see Chapter III of [6]). If $p(x) \in A[x]$, let $p(x)$ denote the coset $p(x) + (\alpha(x))$, and if $\phi \in \Phi_A$, let $p_\phi(x)$ denote $\sum \phi(p_i) x^i$, $p(x) = \sum p x^i$. The space $\Phi_B$ is identifiable with $\{ (\phi, \lambda) \in \Phi_A \times C: \alpha_\lambda(\lambda) = 0 \}$. The action of $(\phi, \lambda) \in \Phi_B$ on $p(x)$ is given by $(\phi, \lambda)(p(x)) = p_\phi(\lambda)$.

Finally, before proceeding to the theorem of this section, we discuss a recent result of B. E. Johnson which will be used in our proof. In the setting of a Banach algebra, he proved in [3] that if $d$ is a derivation of the Banach algebra $A$ into the bounded functions on $\Phi_A$, then the point derivations $d_\phi$ defined by $\phi \circ d$ are continuous on $A$ for all but at most a finite number of $\phi$'s. Now, suppose that at each point $\phi \in \Phi_A$ we have a finite number of point derivations, but no more than $n$, some positive number. Let them be denoted by $d_\phi^1, \ldots, d_\phi^n$, $k \leq n, \phi \in \Phi_A$. If $d_\phi^i(a)$ is bounded for all $i$, $\phi$ and $a \in A$, then by at most $n$ applications of Johnson's result, we have that for all $i$ and for all except possibly a finite number of $\phi$'s, $d_\phi^i$ is continuous on $A$.

Theorem 1. Let $A$ be a semisimple complex algebra and $\alpha(x)$ a monic polynomial of degree $n$ over $A$. If $B = A[x]/(\alpha(x))$ is a Banach algebra with norm $|| \cdot ||$, then $A$ is $|| \cdot ||$-closed in $B$. Consequently, $B$ has unique complete norm topology.

Proof. By the comments preceding the statement of the theorem, we can conclude that each $\phi \in \Phi_A$ is $|| \cdot ||$-continuous on $A$, where $|| \cdot ||$ is the norm on $B$. Thus, $\Phi_A = \Phi_A, \bar{A}$ the $|| \cdot ||$-closure of $A$ in $B$.

Suppose $\{ a_n \}$ is a sequence in $A$ converging to $p(x) \in B$ with respect to $|| \cdot ||$. Then $\phi(a_n) = (\phi, \lambda)(a_n) \to (\phi, \lambda)(p(x)) = \phi(p(\lambda))$ for all $(\phi, \lambda) \in \Phi_B$. This implies that there exists an $a \in A$ such that $\phi(a_n) \to \phi(a)$ for all $\phi \in \Phi_A$. The existence of such an $a$ is shown in the proof of Theorem 3 in [2]. Hence, $(\phi, \lambda)(p(x))$ can be uniquely expressed as the sum of an element in $A$ and an element in the radical $R(B)$ of $B$, namely, $p(x) = a + [p(x) - a]$ and $p(x) - a \in \bar{A} \cap R(B)$.

We next show that there is a finite subset $F$ of $\Phi_B$ such that if $p(x) \in \bar{A}$, $\partial p \leq n - 1$, then $\phi(p^{(j-1)}(\lambda)) = 0$ for $j$ satisfying $2 \leq j \leq M(\phi, \lambda)$ and for all $(\phi, \lambda) \in \Phi_B \setminus F$, where $M(\phi, \lambda)$ denotes the multiplicity of $\lambda$ as a root of $\alpha_\lambda(x) = 0$. For $j \geq 1$, $p^{(j)}(x)$ denotes the $j$th derivative of $p(x)$ and $p^{(0)}(x) = p(x)$. Consider the statement: there exists a finite subset $F \subset \Phi_B$ such that $2 \leq j \leq \operatorname{min}(i, M(\phi, \lambda))$ implies $\phi(p^{(j-1)}(\lambda)) = 0$ for all $p(x) \in \bar{A}$ with $\partial p \leq n - 1$ and for all $(\phi, \lambda) \in \Phi_B \setminus F$. Suppose that
the statement is true for \( i \). We proceed to show that it is true for \( i + 1 \) as well. Define \( d_{(\phi, \lambda)} \) for \((\phi, \lambda) \in \Phi_B\) as follows:

\[
d_{(\phi, \lambda)}(p(x)) = 0 \quad \text{if } M(\phi, \lambda) \leq i \text{ or } (\phi, \lambda) \in F_i,
\]

\[
= \phi(p^{(i)}(\lambda)) \quad \text{otherwise},
\]

where \( p(x) \in A \) and \( \partial p \leq n - 1 \). Clearly, for each \( p(x) \in A \) with \( \partial p \leq n - 1 \),

\[
\sup_{(\phi, \lambda) \in \Phi_B} \left| d_{(\phi, \lambda)}(p(x)) \right| < + \infty,
\]

and \( d_{(\phi, \lambda)} \) is a linear functional on \( A \). To show that \( d_{(\phi, \lambda)} \) is a point derivation at \( \phi \) on \( A \), let \( p(x), q(x) \in A, \partial p \leq n - 1, \partial q \leq n - 1 \) and write \( p(x)q(x) = Q(x)\alpha(x) + r(x) \), where \( Q(x), r(x) \in A[x] \) and \( \partial r \leq n - 1 \). If \( M(\phi, \lambda) \leq i \) or \((\phi, \lambda) \in F_i, \ d_{(\phi, \lambda)} = 0 \) and there is nothing to prove. Suppose, therefore, \( M(\phi, \lambda) > i + 1 \) and \((\phi, \lambda) \in F_i \). Then \( d_{(\phi, \lambda)}(p(x)q(x)) = \phi(r^{(i)}(\lambda)). \) But

\[
r^{(i)}(x) = \sum_{j=0}^{i} \binom{i}{j} \{ p^{(i-j)}(x)q^{(j)}(x) - Q^{(i-j)}(x)\alpha^{(j)}(x) \}.
\]

Thus,

\[
d_{(\phi, \lambda)}(p(x)q(x)) = \sum_{j=0}^{i} \binom{i}{j} \{ \phi(p^{(i-j)}(\lambda))\phi(q^{(j)}(\lambda)) - \phi(Q^{(i-j)}(\lambda))\phi(\alpha^{(j)}(\lambda)) \}
\]

\[
= \phi(p^{(i)}(\lambda))\phi(q(\lambda)) + \phi(p(\lambda))\phi(q^{(i)}(\lambda))
\]

\[
= d_{(\phi, \lambda)}(p(x))(\phi, \lambda)(q(x)) + (\phi, \lambda)(p(x))d_{(\phi, \lambda)}(q(x)),
\]

since \( \phi(\alpha^{(j)}(\lambda)) = 0 \) for \( j = 0, 1, 2, \ldots, i \) and \( p(x), q(x) \in A, \partial p \leq n - 1, \partial q \leq n - 1, (\phi, \lambda) \in F_i \) imply that \( \phi(p^{(i-j)}(\lambda)) = \phi(q^{(i-j)}(\lambda)) = 0 \) for \( j = 2, \ldots, \min(i, M(\phi, \lambda)) = i \). Hence, \( d_{(\phi, \lambda)} \) is a point derivation. By the extension of Johnson’s result mentioned above, we can conclude that there is a finite set \( F_{i+1} \) such that \( d_{(\phi, \lambda)} \) is continuous whenever \((\phi, \lambda) \in \Phi_B \setminus F_{i+1} \). We may and do assume that \( F_{i+1} \supset F_i \). Now, if

\[
a_n \xrightarrow{\| \cdot \|} p(x), \quad \{ a_n \} \subset A,
\]

then \( d_{(\phi, \lambda)}(a_n) = 0 \) for all \( n \) so that \((\phi, \lambda)(p^{(i)}(x)) = 0 \) whenever \( M(\phi, \lambda) \geq i + 1 \) and \((\phi, \lambda) \in F_{i+1} \). Thus \( 2 \leq j \leq \min(i + 1, M(\phi, \lambda)) \) implies that \( \phi(p^{(i-j)}(\lambda)) = 0 \) whenever \((\phi, \lambda) \in F_{i+1} \). Since the argument also shows that the statement is true for \( i = 2 \), we have by induction that \( \phi(p^{(i-j)}(\lambda)) = 0 \) whenever \( 2 \leq j \leq M(\phi, \lambda) \), except possibly for \((\phi, \lambda) \) belonging to a finite subset \( F \) of \( \Phi_B \). (For the case \( i = 2 \), \( d_{(\phi, \lambda)}(p(x)) \) is simply \( \phi(p^{(i)}(\lambda)) \) when \( 2 \leq M(\phi, \lambda) \) and zero otherwise.)
Let $F' = (\pi)^{-1}(\pi(F))$, and let $p(x) \in \bar{A} \cap R(B)$, where $\pi(\phi, \lambda) = \phi$ for $(\phi, \lambda) \in \Phi_B$. Then $(\phi, \lambda) \in \Phi_B$ implies that $\phi(p(\lambda)) = 0$. We may assume that the degree $d$ of $p(x)$ is smaller than $n$. Now, suppose that $(\phi, \lambda) \in F'$ and that $p_\phi(x)$ is not the zero polynomial. Then $\lambda$ is a root of $p_\phi(x) = 0$ of at least multiplicity $M(\phi, \lambda)$. But this implies that $p_\phi(x) = 0$ has at least $n = \sum (\phi, \lambda) \in \Gamma(\phi) M(\phi, \lambda)$ roots, each repeated according to multiplicity. This is a contradiction since $d < n - 1$. Hence $(\phi, \lambda) \in F'$ implies that $p_\phi(x)$ is the zero polynomial. Thus, if $p(x) = \sum_{k=0}^{n-1} p_k x^k \in R(B) \cap \bar{A}$, then each $p_k$ vanishes on $\Phi_A$, except possibly for a finite subset. If $\hat{p}(\phi) \neq 0$, then $\phi$ is an isolated point of $\Phi_A$.

We now continue as in [2]. If $\phi$ is an isolated point in $\Phi_A$, then there exists an idempotent $u_\phi$ in $A$ such that $u_\phi(\theta) = 1$ if and only if $\theta = \phi$ (see [6, p. 168]), since $A$ is a Banach algebra under some norm (see [2, Lemma 1]). Since $A$ is semisimple, $u_\phi a = \phi(a) u_\phi$, $a \in A$, and therefore $u_\phi A$ is closed in $B$. Now, if $p(x) = p_0 + \cdots + p_{n-1} x^{n-1} \in B$, then $u_\phi p(x) = \phi(p_0) u_\phi + \phi(p_1) u_\phi x + \cdots + \phi(p_{n-1}) u_\phi x^{n-1}$. Furthermore if

$$a_n \rightarrow p(x),$$

then $u_\phi a_n \rightarrow u_\phi p(x)$ and $u_\phi p(x)$ must lie in $u_\phi A$. Hence, $u_\phi p_k = 0$, $1 \leq k \leq n - 1$, or equivalently, $\phi(p_k) = 0$ for $1 \leq k \leq n - 1$. Thus, $p(x) \in \bar{A}$ implies that $p(x) = p_0 \in A$ and $A = \bar{A}$ since $A$ is semisimple.

The second assertion follows from Theorem 2 of [2].

2. Derivations on $A[x]/(\alpha(x))$. In our context, the work of Loy in [4] shows that if $A$ is a semisimple Banach algebra, then any derivation $D$ on $B = A[x]/(\alpha(x))$ must be of the form $D(p(x)) = p'(x) D(x)$, $p(x) = p_0 + p_1 x + \cdots + p_{n-1} x^{n-1}$, $n = d \alpha$. This particularly simple form reflects the fact that $A$ has no nontrivial derivations into itself [3]. It is easily established that any linear transformation on $B$ of the form $p(x) \rightarrow p'(x) p_0(x)$, $p_0(x)$ fixed, is a derivation on $B$ if and only if $\alpha'(x) p_0(x) = 0$. Note that $p(x) \rightarrow p'(x) p_0(x)$ is continuous with respect to any Banach algebra norm on $B$, by virtue of Theorem 1.

Theorem 2. Let $A$ be a semisimple commutative Banach algebra and $\alpha(x) \in A[x]$ a monic polynomial. Then $B$ has a nontrivial derivation if and only if the radical of $B$ is nontrivial.

Proof. The only if part follows from Theorem 4 of [3].

Suppose, now, that $R(B) \neq (0)$. By Corollary 9.3 of [5], we have that the discriminant $d_\alpha$ of $\alpha(x)$ is either zero or a zero divisor in $A$, say $d_\alpha c = 0$, $c \neq 0$. From Lemma 9.1 of [5], we have the existence of
polynomials $\gamma(x)$, $\delta(x)$, $R(x) \in A[x]$ and nonzero elements $a, b \in A$ such that

(i) $aa'(x) = \gamma(x)R(x),$
(ii) $ba'(x) = \delta(x)R(x),$
(iii) $\phi(a) = 0$ if and only if $\phi(b) = 0$, and $\phi(c) = 0$ implies $\phi(a) = 0$, and
(iv) if $\beta'(x) = \alpha'(x)$, $\beta(x) \mid \alpha(x)$, then $\beta(x) \mid R(x)$, where $\phi \in \Phi_A$.

Set $p_0(x) = ab\gamma(x)$, and $D(p(x)) = p'(x)p_0(x)$. To show that $D$ is a derivation, we need only verify that $\alpha'(x)p_0(x) = 0$, that is, $\alpha'(x)p_0(x) \in (\alpha(x))$. By conditions (i) and (ii), we have that

$$p_0(x)\alpha'(x) = ab\gamma(x)\alpha'(x) = a\gamma(x)\delta(x)R(x) = a^2\delta(x)\alpha(x).$$

We next show that $D$ is nontrivial.

Suppose that $p_0(x) = q(x)\alpha(x)$. Then $q(x)\alpha'(x)\alpha(x) = a^2\delta(x)\alpha(x)$. Since $\alpha(x)$ is monic and hence not a zero divisor in $A[x]$, $q(x)\alpha'(x) = a^2\delta(x)$. Now, $a^2b\alpha'(x) = a^2\delta(x)R(x) = q(x)\alpha'(x)R(x)$. Hence, $a^2b = q(x)R(x)$. If $\phi \in \Phi_A$ and $\phi(a) \neq 0$, then $q_\phi(x)R_\phi(x) = \phi(a^2b) \neq 0$ by (iii), and consequently, $q_\phi(x)$ and $R_\phi(x)$ are constant polynomials. On the other hand, if $\phi \in \Phi_A$ and $\phi(a) \neq 0$, then $\phi(c) \neq 0$. Therefore, $\phi(d_a) = 0$ so that there exists a complex number $\lambda$ such that $(x-\lambda)\mid \alpha_\phi(x)$ and $(x-\lambda)\mid \alpha'_\phi(x)$. By (iv), $(x-\lambda)\mid R_\phi(x)$ so that $\partial R_\phi(x) > 0$, a contradiction. It follows that $p_0(x) \in (\alpha(x))$, and $D$ as defined is not trivial. This completes the proof.

In conclusion, we note that since the results from [5] used above are valid for semisimple complex algebras, Theorem 2 is true for semisimple complex algebras $A$ without nontrivial derivations.

**References**


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