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A CLASS OF COMMUTATIVE BANACH ALGEBRAS WITH UNIQUE COMPLETE NORM TOPOLOGY AND CONTINUOUS DERIVATIONS

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ABSTRACT. Let A be a semisimple commutative complex algebra with identity and $\alpha(x)$ a monic polynomial over A. Two results are proved. If $B=A[x]/(\alpha(x))$ is a Banach algebra under some norm, then B has a unique complete norm topology. Furthermore, B has nontrivial derivations if and only if B has a nontrivial radical.

Introduction. It is known that if A is a commutative Banach algebra with identity and $\alpha(x)$ is a monic polynomial over A, then $B=A[x]/(\alpha(x))$ can be made into a Banach algebra with a norm extending the norm on A [1]. In [2], D. T. Brown has shown that if A is semisimple and regular (in the sense of Šilov), then B has unique complete norm topology. The assumption of regularity can be dropped as we show in §1. In general, the extension B is not semisimple; indeed, B has a nontrivial radical if and only if the discriminant of $\alpha(x)$ is zero or a zero divisor in A [5].

§2 is motivated by a recent result of R. J. Loy that states, in our context, that all derivations on B are continuous [4]. However, he does not show the existence of derivations on (nonsemisimple) B even though he gives their form. We consider the existence of nontrivial derivations in §2.

1. **Uniqueness of complete norm topology on** $A[x]/(\alpha(x))$. It is convenient to set first some of our notation and to make several comments. Lemma 1 of [2] states that if A is a semisimple algebra (all algebras in this paper are assumed to be over the complex field C and to possess an identity element) and if $B=A[x]/(\alpha(x))$ is a Banach algebra, then A is a Banach algebra under some norm. Actually, it is proved that if $K(A)$ denotes the intersection of the kernels of nontrivial complex homomorphisms, then $A/K(A)$ is a Banach algebra. However, it is known that each maximal ideal M in A is contained in a maximal ideal M_0 in B such that $M=M_0\cap A$ (see p. 259 of [7]). Since B is a Banach algebra, M_0 is the kernel of a com-

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plex homomorphism; hence, so is M. Thus, $K(A)=(0)$ since A is semisimple and Brown's Lemma 1 follows.

For a complex algebra A, let Φ_A denote the space of nontrivial complex homomorphisms on A, together with the Gelfand topology (see Chapter III of [6]). If $p(x)\in A[x]$, let $p(\mathbf{x})$ denote the coset $p(x)+(\alpha(x))$, and if $\phi\in\Phi_A$, let $p_\phi(x)$ denote $\sum \phi(p_i)x^i$, $p(x)=\sum p_i x^i$. The space Φ_B is identifiable with $\{(\phi, \lambda)\in\Phi_A\times C:\alpha_\phi(\lambda)=0\}$. The action of $(\phi, \lambda)\in\Phi_B$ on $p(\mathbf{x})$ is given by $(\phi, \lambda)(p(\mathbf{x}))=p_\phi(\lambda)$.

Finally, before proceeding to the theorem of this section, we discuss a recent result of B. E. Johnson which will be used in our proof. In the setting of a Banach algebra, he proved in [3] that if d is a derivation of the Banach algebra A into the bounded functions on Φ_A, then the point derivations d_ϕ defined by $\phi\circ d$ are continuous on A for all but at most a finite number of ϕ's. Now, suppose that at each point $\phi\in\Phi_A$ we have a finite number of point derivations, but no more than n, some positive number. Let them be denoted by $d_\phi^1, \cdots, d_\phi^{k_\phi}$, $k_\phi\leqq n$, $\phi\in\Phi_A$. If $d_\phi^i(a)$ is bounded for all i, ϕ and $a\in A$, then by at most n applications of Johnson's result, we have that for all i and for all except possibly a finite number of ϕ's, d_ϕ^i is continuous on A.

THEOREM 1. *Let A be a semisimple complex algebra and $\alpha(x)$ a monic polynomial of degree n over A. If $B=A[x]/(\alpha(x))$ is a Banach algebra with norm $\|\cdot\|$, then A is $\|\cdot\|$-closed in B. Consequently, B has unique complete norm topology.*

PROOF. By the comments preceding the statement of the theorem, we can conclude that each $\phi\in\Phi_A$ is $\|\cdot\|$-continuous on A, where $\|\cdot\|$ is the norm on B. Thus, $\Phi_{\bar{A}}=\Phi_A$, $\bar{A}$ the $\|\cdot\|$-closure of A in B.

Suppose $\{a_n\}$ is a sequence in A converging to $p(\mathbf{x})\in B$ with respect to $\|\cdot\|$. Then $\phi(a_n)=(\phi, \lambda)(a_n)\to(\phi, \lambda)(p(\mathbf{x}))=\phi(p(\lambda))$ for all $(\phi, \lambda)\in\Phi_B$. This implies that there exists an $a\in A$ such that $\phi(a_n)\to\phi(a)$ for all $\phi\in\Phi_A$. The existence of such an a is shown in the proof of Theorem 3 in [2]. Hence, $p(\mathbf{x})$ can be uniquely expressed as the sum of an element in A and an element in the radical $R(B)$ of B, namely, $p(\mathbf{x})=a+[p(\mathbf{x})-a]$ and $p(\mathbf{x})-a\in\bar{A}\cap R(B)$.

We next show that there is a finite subset F of Φ_B such that if $p(\mathbf{x})\in\bar{A}$, $\partial p\leqq n-1$, then $\phi(p^{(j-1)}(\lambda))=0$ for j satisfying $2\leqq j\leqq M(\phi, \lambda)$ and for all $(\phi, \lambda)\in\Phi_B\backslash F$, where $M(\phi, \lambda)$ denotes the multiplicity of λ as a root of $\alpha_\phi(x)=0$. For $j\geqq 1$, $p^{(j)}(x)$ denotes the jth derivative of $p(x)$ and $p^{(0)}(x)=p(x)$. Consider the statement: there exists a finite subset $F_i\subset\Phi_B$ such that $2\leqq j\leqq\min(i, M(\phi, \lambda))$ implies $\phi(p^{(j-1)}(\lambda))=0$ for all $p(\mathbf{x})\in\bar{A}$ with $\partial p\leqq n-1$ and for all $(\phi, \lambda)\in\Phi_B\backslash F_i$. Suppose that

the statement is true for i. We proceed to show that it is true for $i+1$ as well. Define $d_{(\phi,\lambda)}$ for $(\phi, \lambda)\in\Phi_B$ as follows:

$$\begin{aligned} d_{(\phi,\lambda)}(p(\mathbf{x})) &= 0 && \text{if } M(\phi, \lambda) \leqq i \text{ or } (\phi, \lambda) \in F_i, \\ &= \phi(p^{(i)}(\lambda)) && \text{otherwise,} \end{aligned}$$

where $p(\mathbf{x})\in\overline{A}$ and $\partial p\leqq n-1$. Clearly, for each $p(\mathbf{x})\in\overline{A}$ with $\partial p\leqq n-1$,

$$\sup_{(\phi,\lambda)\in\Phi_B} \left| d_{(\phi,\lambda)}(p(\mathbf{x})) \right| < +\infty,$$

and $d_{(\phi,\lambda)}$ is a linear functional on $\overline{A}$. To show that $d_{(\phi,\lambda)}$ is a point derivation at ϕ on $\overline{A}$, let $p(\mathbf{x})$, $q(\mathbf{x})\in\overline{A}$, $\partial p\leqq n-1$, $\partial q\leqq n-1$ and write $p(x)q(x)=Q(x)\alpha(x)+r(x)$, where $Q(x), r(x)\in A[x]$ and $\partial r\leqq n-1$. If $M(\phi, \lambda)\leqq i$ or $(\phi, \lambda)\in F_i$, $d_{(\phi,\lambda)}=0$ and there is nothing to prove. Suppose, therefore, $M(\phi, \lambda)\geqq i+1$ and $(\phi, \lambda)\notin F_i$. Then $d_{(\phi,\lambda)}(p(\mathbf{x})q(\mathbf{x}))=\phi(r^{(i)}(\lambda))$. But

$$r^{(i)}(x) = \sum_{j=0}^{i}\binom{i}{j}\left\{p^{(i-j)}(x)q^{(j)}(x) - Q^{(i-j)}(x)\alpha^{(j)}(x)\right\}.$$

Thus,

$$\begin{aligned} d_{(\phi,\lambda)}(p(\mathbf{x})q(\mathbf{x})) &= \sum_{j=0}^{i}\binom{i}{j}\left\{\phi(p^{(i-j)}(\lambda))\phi(q^{(j)}(\lambda))-\phi(Q^{(i-j)}(\lambda))\phi(\alpha^{(i)}(\lambda))\right\} \\ &= \phi(p^{(i)}(\lambda))\phi(q(\lambda)) + \phi(p(\lambda))\phi(q^{(i)}(\lambda)) \\ &= d_{(\phi,\lambda)}(p(\mathbf{x}))(\phi, \lambda)(q(\mathbf{x})) + (\phi, \lambda)(p(\mathbf{x}))d_{(\phi,\lambda)}(q(\mathbf{x})), \end{aligned}$$

since $\phi(\alpha^{(j)}(\lambda))=0$ for $j=0, 1, 2, \cdots, i$ and $p(\mathbf{x})$, $q(\mathbf{x})\in\overline{A}$, $\partial p\leqq n-1$, $\partial q\leqq n-1$, $(\phi, \lambda)\notin F_i$ imply that $\phi(p^{(j-1)}(\lambda))=\phi(q^{(j-1)}(\lambda))=0$ for $j=2, \cdots, \min(i, M(\phi, \lambda))=i$. Hence, $d_{(\phi,\lambda)}$ is a point derivation. By the extension of Johnson's result mentioned above, we can conclude that there is a finite set F_{i+1} such that $d_{(\phi,\lambda)}$ is continuous whenever $(\phi, \lambda)\in\Phi_B\backslash F_{i+1}$. We may and do assume that $F_{i+1}\supset F_i$. Now, if

$$a_n \xrightarrow{\|\cdot\|} p(\mathbf{x}), \qquad \{a_n\} \subset A,$$

then $d_{(\phi,\lambda)}(a_n)=0$ for all n so that $(\phi, \lambda)(p^{(i)}(\mathbf{x}))=0$ whenever $M(\phi, \lambda)\geqq i+1$ and $(\phi, \lambda)\notin F_{i+1}$. Thus $2\leqq j\leqq\min(i+1, M(\phi, \lambda))$ implies that $\phi(p^{(j-1)}(\lambda))=0$ whenever $(\phi, \lambda)\notin F_{i+1}$. Since the argument also shows that the statement is true for $i=2$, we have by induction that $\phi(p^{(j-1)}(\lambda))=0$ whenever $2\leqq j\leqq M(\phi, \lambda)$, except possibly for (ϕ, λ) belonging to a finite subset F of Φ_B. (For the case $i=2$, $d_{(\phi,\lambda)}(p(\mathbf{x}))$ is simply $\phi(p^{(1)}(\lambda))$ when $2\leqq M(\phi, \lambda)$ and zero otherwise.)

Let $F' = (\pi)^{-1}(\pi(F))$, and let $p(\mathrm{x}) \in \overline{A} \cap R(B)$, where $\pi(\phi, \lambda) = \phi$ for $(\phi, \lambda) \in \Phi_B$. Then $(\phi, \lambda) \in \Phi_B$ implies that $\phi(p(\lambda)) = 0$. We may assume that the degree ∂p of $p(x)$ is smaller than n. Now, suppose that $(\phi, \lambda) \notin F'$ and that $p_\phi(x)$ is not the zero polynomial. Then λ is a root of $p_\phi(x) = 0$ of at least multiplicity $M(\phi, \lambda)$. But this implies that $p_\phi(x) = 0$ has at least $n = \sum_{(\phi,\lambda)\in(\pi)^{-1}(\phi)} M(\phi, \lambda)$ roots, each repeated according to multiplicty. This is a contradiction since $\partial p \leqq n-1$. Hence $(\phi, \lambda) \notin F'$ implies that $p_\phi(x)$ is the zero polynomial. Thus, if $p(\mathrm{x}) = \sum_{k=0}^{n-1} p_k \mathrm{x}^k \in R(B) \cap \overline{A}$, then each $\hat{p}_k$ vanishes on Φ_A, except possibly for a finite subset. If $\hat{p}_k(\phi) \neq 0$, then ϕ is an isolated point of Φ_A.

We now continue as in [2]. If ϕ is an isolated point in Φ_A, then there exists an idempotent u_ϕ in A such that $\hat{u}_\phi(\theta) = 1$ if and only if $\theta = \phi$ (see [6, p. 168]), since A is a Banach algebra under some norm (see [2, Lemma 1]). Since A is semisimple $u_\phi a = \phi(a) u_\phi$, $a \in A$, and therefore $u_\phi A$ is closed in B. Now, if $p(\mathrm{x}) = p_0 + \cdots + p_{n-1}\mathrm{x}^{n-1} \in B$, then $u_\phi p(\mathrm{x}) = \phi(p_0) u_\phi + \phi(p_1) u_\phi \mathrm{x} + \cdots + \phi(p_{n-1}) u_\phi \mathrm{x}^{n-1}$. Furthermore if

$$a_n \xrightarrow{\|\cdot\|} p(\mathrm{x}),$$

then $u_\phi a_n \to u_\phi p(\mathrm{x})$ and $u_\phi p(\mathrm{x})$ must lie in $u_\phi A$. Hence, $u_\phi p_k = 0$, $1 \leqq k \leqq n-1$, or equivalently, $\phi(p_k) = 0$ for $1 \leqq k \leqq n-1$. Thus, $p(\mathrm{x}) \in \overline{A}$ implies that $p(\mathrm{x}) = p_0 \in A$ and $A = \overline{A}$ since A is semisimple.

The second assertion follows from Theorem 2 of [2].

2. **Derivations on** $A[x]/(\alpha(x))$. In our context, the work of Loy in [4] shows that if A is a semisimple Banach algebra, then any derivation D on $B = A[x]/(\alpha(x))$ must be of the form $D(p(\mathrm{x})) = p'(\mathrm{x}) D(\mathrm{x})$, $p(x) = p_0 + p_1 x + \cdots + p_{n-1} x^{n-1}$, $n = \partial\alpha$. This particularly simple form reflects the fact that A has no nontrivial derivations into itself [3]. It is easily established that any linear transformation on B of the form $p(\mathrm{x}) \to p'(\mathrm{x}) p_0(\mathrm{x})$, $p_0(\mathrm{x})$ fixed, is a derivation on B if and only if $\alpha'(\mathrm{x}) p_0(\mathrm{x}) = 0$. Note that $p(\mathrm{x}) \to p'(\mathrm{x}) p_0(\mathrm{x})$ is continuous with respect to any Banach algebra norm on B, by virtue of Theorem 1.

THEOREM 2. *Let A be a semisimple commutative Banach algebra and $\alpha(x) \in A[x]$ a monic polynomial. Then B has a nontrivial derivation if and only if the radical of B is nontrivial.*

PROOF. The only if part follows from Theorem 4 of [3].

Suppose, now, that $R(B) \neq (0)$. By Corollary 9.3 of [5], we have that the discriminant d_α of $\alpha(x)$ is either zero or a zero divisor in A, say $d_\alpha c = 0$, $c \neq 0$. From Lemma 9.1 of [5], we have the existence of

polynomials $\gamma(x)$, $\delta(x)$, $R(x)\in A[x]$ and nonzero elements a, $b\in A$ such that

(i) $a\alpha(x)=\gamma(x)R(x)$,

(ii) $b\alpha'(x)=\delta(x)R(x)$,

(iii) $\phi(a)=0$ if and only if $\phi(b)=0$, and $\phi(c)=0$ implies $\phi(a)=0$, and

(iv) if $\beta_\phi(x)\,|\,\alpha_\phi(x)$, $\beta_\phi(x)\,|\,\alpha'_\phi(x)$, then $\beta_\phi(x)\,|\,R_\phi(x)$, where $\phi\in\Phi_A$.

Set $p_0(x)=ab\gamma(x)$, and $D(p(x))=p'(x)p_0(x)$. To show that D is a derivation, we need only verify that $\alpha'(x)p_0(x)=0$, that is, $\alpha'(x)p_0(x)\in(\alpha(x))$. By conditions (i) and (ii), we have that

$$p_0(x)\alpha'(x) = ab\gamma(x)\alpha'(x) = a\gamma(x)\delta(x)R(x) = a^2\delta(x)\alpha(x).$$

We next show that D is nontrivial.

Suppose that $p_0(x)=q(x)\alpha(x)$. Then $q(x)\alpha'(x)\alpha(x)=a^2\delta(x)\alpha(x)$. Since $\alpha(x)$ is monic and hence not a zero divisor in $A[x]$, $q(x)\alpha'(x)=a^2\delta(x)$. Now, $a^2b\alpha'(x)=a^2\delta(x)R(x)=q(x)\alpha'(x)R(x)$. Hence, $a^2b=q(x)R(x)$. If $\phi\in\Phi_A$ and $\phi(a)\neq 0$, then $q_\phi(x)R_\phi(x)=\phi(a^2b)\neq 0$ by (iii), and consequently, $q_\phi(x)$ and $R_\phi(x)$ are constant polynomials. On the other hand, if $\phi\in\Phi_A$ and $\phi(a)\neq 0$, then $\phi(c)\neq 0$. Therefore, $\phi(d_\alpha)=0$ so that there exists a complex number λ such that $(x-\lambda)\,|\,\alpha_\phi(x)$ and $(x-\lambda)\,|\,\alpha'_\phi(x)$. By (iv), $(x-\lambda)\,|\,R_\phi(x)$ so that $\partial R_\phi(x)>0$, a contradiction. It follows that $p_0(x)\notin(\alpha(x))$, and D as defined is not trivial. This completes the proof.

In conclusion, we note that since the results from [5] used above are valid for semisimple complex algebras, Theorem 2 is true for semisimple complex algebras A without nontrivial derivations.

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