A ZERO-ONE LAW FOR GAUSSIAN PROCESSES

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Abstract. Let \( P_0 \) be a Gaussian probability measure on the measurable space \((X, B(X))\), where \( X \) is a linear space of real-valued functions over a complete separable metric space \( T \), and \( B(X) \) is the \( \sigma \)-algebra generated by sets of the form \( \{x \in X: (x(t_1), \ldots, x(t_n)) \in B^n \} \); \( B^n \) being the Borel sets of \( \mathbb{R}^n \), \( n \geq 1 \). The covariance \( R(s, t) \) is assumed continuous on \( T \times T \). If \( G \) is a subgroup of \( X \) and belongs to the \( \sigma \)-algebra \( B_0(X) \) (the completion of \( B(X) \) with respect to \( P_0 \)), then it is shown that \( P_0(G) = 0 \) or 1.

1. Introduction. For the background and history of the problem we refer to [1] and [2]. We would like to point out here that Jamison and Orey in [1] proved the above result for the special case where the Gaussian process involved had continuous paths. Kallianpur [2] proved such a result for \( \mathbb{R} \)-modules (groups closed under multiplication by rationals). Kallianpur’s result for groups is restricted to those which are \( B(X) \)-measurable rather than \( B_0(X) \)-measurable. He points out in [2] why his proof does not work for \( B_0(X) \)-measurable subgroups. Our main result (Theorem 1) unifies and generalizes the results of [1] and [2], and also gives an answer in the affirmative to the conjecture made in [1]. Our method of proof is similar to the one given in [2], but it is simpler. The notation used here is also essentially the same as in [2].

\( T \) is a complete separable metric space. \( X \) is a linear space of real-valued functions defined on \( T \) with the usual operation of addition of functions and multiplication by real scalars. \( B(X) \) is the \( \sigma \)-algebra as explained above. \( P_0 \) is a Gaussian measure on \((X, B(X))\). We assume that

\[
\int_X x(t) P_0(dx) = 0 \quad \text{for each } t \in T; \\
\int_X x(s)x(t) P_0(dx) = R(s, t), \quad s, t \in T,
\]

is continuous on \( T \times T \). \( H(R) \) will denote the reproducing kernel Hilbert space of the covariance \( R \). It is also assumed that \( H(R) \subset X \).

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$B_0(X)$ denotes the completion of the $\sigma$-algebra $B(X)$ with respect to $P_0$.

The main result is the following:

**Theorem 1.** Let $G$ be a $B_0(X)$-measurable subgroup of the linear space $X$. Then $P_0(G) = 0$ or $1$.

**Remark.** For applications of this result we refer to Lemma 5 [2].

2. Proof of Theorem 1. We need the following lemmas.

**Lemma 1.** Let $F \in B_0(X)$. If $m \in H(R)$ and $\alpha$ is a real number, then the set

$$F_\alpha = \{ x \in X : x = y + \alpha m, \ y \in F \}$$

is in $B_0(X)$, and

$$\lim_{\alpha \to 0} P_0(F_\alpha) = P_0(F). \tag{2.1}$$

**Proof.** This is established in the proof of Lemma 5 [2].

**Lemma 2.** Let $\{ e_j \}_{1}^\infty$ be a complete orthonormal system in $H(R)$ and $g$ be a $B_0(X)$-measurable real function such that for each $x \in X$ and every rational $r$, $g(x + re_j) = g(x)$, $j = 1, 2, \ldots$. Then $g(x) = \text{constant \ a.s.} \ (P_0)$.

**Proof.** See Lemma 6 [2].

To finish the proof of the theorem, let $P_0(G) > 0$. We will show that this implies that $H(R) \subseteq G$. It is then seen easily that $P_0(G) = 1$ as follows: since $G$ is a group and $H(R)$ a Hilbert space with a complete orthonormal system $\{ e_j \}$, it follows that $x \in G$ if and only if $x + re_j \in G$ for every rational $r$, $j = 1, 2, \ldots$. Let $I_0$ be the indicator function of the set $G$, then by Lemma 2 we have $I_0 = \text{constant \ a.s.} \ (P_0)$. But this constant must be equal to 1 since $P_0(G) > 0$.

It thus remains to show that $P_0(G) > 0$ implies that $H(R) \subseteq G$. There exists a positive integer $s$ such that $P_0(G) > 1/s$. We hold this $s$ fixed. Let $m \in H(R)$, $m \in G$. This will lead to a contradiction. For each positive integer $n$ define $s + 1$ sets $G_0^{(n)}, G_1^{(n)}, \ldots, G_s^{(n)}$ as follows:

$$G_0^{(n)} = G, \ G_k^{(n)} = \{ x : x = y + (s!kn)^{-1}m, \ y \in G \}, \ 1 \leq k \leq s. \tag{2.2}$$

By Lemma 1 these sets are in $B_0(X)$ and

$$\lim_{n} P_0(G_k^{(n)}) = P_0(G), \quad 0 \leq k \leq s. \tag{2.3}$$

We now show that these sets must be pairwise disjoint. If $G_0^{(n)}$ and
$G^{(n)}_k$, $k > 0$, have an element in common, then for some $x, y \in G$, $x = y + (s!kn)^{-1}m$; hence $m = (s!kn)(x - y)$, an element of $G$, which contradicts that $m \notin G$. Suppose now that $G^{(n)}_j$, $G^{(n)}_k$, $1 \leq j < k$, have an element in common. Then for some $x, y \in G$,

$$x + (s!jn)^{-1}m = y + (s!kn)^{-1}m.$$  

Hence $m = (s!kjn)(k - j)^{-1}(y - x)$. But $(s!kjn)(k - j)^{-1}$ is a positive integer and this implies again that $m \in G$, a contradiction. We thus have $\sum_{k=0}^{n} P_0(G^{(n)}_k) \leq 1$, $n = 1, 2, \ldots$. Letting $n$ tend to infinity, we conclude from (2.3) that $(s+1)P_0(G) \leq 1$. Since $P_0(G) > 1/s$, this is a contradiction. Hence $H(R) \subset G$ and the proof is complete.

References


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