PROPERTY P AND DIRECT INTEGRAL DECOMPOSITION OF $W^*$ ALGEBRAS

PAUL WILLIG

Abstract. If $\mathfrak{A}$ is a $W^*$ algebra on separable Hilbert space $H$, and if $\mathfrak{A}(\lambda)$ are the factors in the direct integral decomposition of $\mathfrak{A}$, then $\mathfrak{A}^0 = \{x | \mathfrak{A}(\lambda) \text{ has property P} \}$ is $\mu$-measurable, and $\mathfrak{A}$ has property P $\mu$-a.e.

Let $\mathfrak{A}$ be a $W^*$ algebra on separable Hilbert space $H$. Let $\mathfrak{A}$ have direct integral decomposition into factors

$$\mathfrak{A} = \int_{\Lambda} \bigoplus \mathfrak{A}(\lambda) \mu(d\lambda),$$

where $K$ denotes the underlying separable Hilbert space of $H$. We assume that the reader is familiar with the notation and methods of [7, Chapter 1] and [8].

We establish the following notation for this paper. $\mathfrak{A}_1$ denotes the unit sphere of $\mathfrak{A}$, and $\mathfrak{U}(\mathfrak{A})$ denotes the unitary operators in $\mathfrak{A}$. $Z$ will denote the center of $\mathfrak{A}$ ($Z$ consists of all diagonal operators in $B(H)$ [7, Theorem 1.5.9]). $\mathfrak{E}(\mathfrak{A})$ denotes the set of all nonnegative real-valued functions on $\mathfrak{U}(\mathfrak{A})$ which vanish except at a finite number of points and which satisfy $\sum_{U \in \mathfrak{U}(\mathfrak{A})} f(U) = 1$. We call the finite set of $U$ such that $f(U) \neq 0$ the support of $f$. We write $f \cdot T = \sum_{U \in \mathfrak{U}(\mathfrak{A})} f(U) U T U^*$ for $T \in B(H)$ (or $T \in B(K)$ if we are dealing with $\mathfrak{A}(\lambda)$).

If $L$ is a separable Hilbert space, let $S$ denote the unit sphere of $L$ and let $\{x_i\}$ be a fixed dense sequence in $S$. If $d$ denotes the metric of [7, Lemma 1.4.8] which defines the weak topology on bounded sets in $B(L)$, then, defining $W(A) = d(A, 0)$, we have, for a bounded sequence $T_n$, that $T_n \to 0$ weakly iff $W(T_n) \to 0$. Moreover, if $U$ is unitary, $W_U(A) = W(UA U^*)$ also determines weak convergence to 0. We use this fact below in Lemma 3.

It follows from [8, Lemma 1.5] and from the proof of [8, Lemma 3.5] that there are countable sequences $A_n \subseteq A_1 \subseteq A'_1 \subseteq \mathfrak{A}_1$, $U_n \subseteq \mathfrak{U}(\mathfrak{A})$ such that for $\mu$-a.e. $\lambda$ the sequence $A_n(\lambda)$ ($A'_n(\lambda)$, $U_n(\lambda)$) is strong-* dense in $\mathfrak{A}(\lambda)$, $\mathfrak{A}'(\lambda)$, $\mathfrak{U}(\mathfrak{A}(\lambda))$. Moreover (see remark after [8,
Lemma 2.2]) we may assume that all operators we deal with are strong-* continuous.

Our aim in this paper is to study property P of \( \alpha \) in relation to the factors \( \alpha(\lambda) \).

Definition 1 [7, p. 168]. \( \alpha \) has property P (property CP) if for every \( T \in B(H) \) (every \( T \in Z' \)) the intersection of the weakly closed convex hull of \( K(T) = \{ UTU^* | U \in \mathcal{U}(\alpha) \} \) with \( \alpha' \) is not empty.

For a factor, property CP is simply property P. We shall show that this is always the case, and we use this to prove that \( \alpha \) has property P iff \( \alpha(\lambda) \) has property P \( \mu \)-a.e.

Lemma 2. Let \( \mathcal{B} \) be an Abelian \( W* \)-algebra on separable Hilbert space \( L \). Then \( \mathcal{B} \) has property P.

Proof. By [7, Lemma 2.1] \( \mathcal{B} \) is generated by a Hermitian operator \( A \). We may assume that \( \sigma(A) \subset [0, 1] \). Letting \( E_t \) denote the spectral projections of \( A \), it is clear that \( \mathcal{B} \) is generated by the increasing sequence of finite-dimensional \(*\)-subalgebras \( \mathcal{B}_m \) generated by \( \{ E_t | t = n2^{-m}, n = 1, 2, \ldots, 2^m \} \). Hence \( \mathcal{B} \) has property P [7, p. 168]. Q.E.D.

Lemma 3. If \( \mathcal{B} \) has property CP then \( \alpha \) has property P.

Proof. Suppose \( \mathcal{B} \) has property CP. Let \( T \in B(H) \) be given. Since we are dealing with weak convergence on bounded sets of operators, it follows from Lemma 2 (with \( \mathcal{B} = \mathcal{Z} \) and \( H = L \)) and our hypothesis that there are sequences \( f_k \in \mathcal{B}(\mathcal{Z}) \) and \( g_k \in \mathcal{B}(\alpha) \) and operators \( A \in Z' \) and \( A' \in \alpha' \) such that \( f_k \cdot T \to A \) weakly and \( g_k \cdot A \to A' \) weakly. Define \( W \) as above for \( L = H \), and let \( B_k = f_k \cdot T \) and \( C_k = g_k \cdot A \). Then \( W(B_k - A) \to 0 \) and \( W(C_k - A') \to 0 \). It suffices to show that given \( \epsilon > 0 \) there is \( h \in \mathcal{B}(\alpha) \) such that \( W(h \cdot T - A') < \epsilon \) in order to conclude that \( \alpha \) has property P.

Given \( \epsilon > 0 \), we can find \( g_k \) such that \( W(C_k - A') < \epsilon/2 \). Let the support of \( g_k \) consist of \( m \) operators. Then we can choose \( f_k \) so that, for each \( U \) in the support of \( g_k \), \( W(U(B_k - A)) < \epsilon/(2m) \). It follows that

\[
W((g_k \cdot B_k) - A') < \epsilon.
\]

Moreover, if we put \( h = g_k \ast f_k \) (convolution) then \( g_k \cdot B_k = h \cdot T \) and \( h \in \mathcal{B}(\alpha) \). Q.E.D.

We now come to our main results. In the remaining portion of this paper, \( W \) is defined as above for \( L = K \), and \( \mathcal{S} \) denotes the unit sphere in \( B(K) \).

Theorem 4. \( \varphi = \{ \lambda | \alpha(\lambda) \text{ has property P} \} \) is \( \mu \)-measurable.
Proof. We shall give a measurable characterization of the set \( \varphi' = \Delta - \varphi \). We begin by considering a \( W^* \)-algebra \( B \) on \( K \) which has property \( P \). Given \( T \in B \), it follows that there are \( f_k \in \mathcal{E}(B) \) and \( B' \in \mathcal{B}' \) such that \( f_k \cdot T = T_k \mapsto B' \) weakly. Since bounded sets are weakly compact and since \( W \) determines weak convergence on bounded sets, this is equivalent to the statement that, if \( B'_n \) are dense in \( \mathcal{B}' \), then there are \( B'_n \) such that \( W(T_k - B'_n) \to 0 \). Next, suppose \( \{U_n\} \) are dense in \( U(\mathcal{B}) \). Then it is easy to see that we may assume that the support of each \( f_k \) is contained in \( \{U_n\} \) and that the values \( f(U_n) \) are rational. This is the key to our theorem.

Let \( \mathcal{F} \) be that subset of \( \mathcal{E}(\mathfrak{A}) \) consisting of \( f \) with support contained in \( \{U_n\} \) and whose values \( f(U_n) \) are rational. Clearly \( \mathcal{F} \) is countable. For each \( f \in \mathcal{F} \) and each pair of positive integers \( (k, m) \) define a subset \( E(f, k, m) \) of \( \Delta \times \mathcal{B} \) consisting of pairs \( (\lambda, T) \) satisfying the following condition, where by \( (f \cdot T)(\lambda) \) we mean \( \sum_{\mathcal{U}_n} f(U_n) \mathcal{U}_n(\lambda) T(U_n) \mathcal{U}_k \).

\[
W((f \cdot T)(\lambda) - A_k(\lambda)) \leq 1/m.
\]

Each set \( E(f, k, m) \) is closed. It follows from our remarks above that, if we let \( \pi \) denote the projection of \( \Delta \times \mathcal{B} \) onto \( \Delta \), then \( \varphi' \) differs by a \( \mu \)-null set from the set

\[
E = \pi \left( \bigcup_{m=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{f \in \mathcal{F}} E(f, k, m) \right).
\]

\( E \) is \( \mu \)-measurable by [7, Lemma 1.4.6] and the theorem follows.

Q.E.D.

Theorem 5. \( \mathfrak{A} \) has property \( P \) iff \( \mu(P') = 0 \).

Proof. Suppose \( \mu(P') > 0 \). Then it follows from our last proof and from [7, Lemma 1.4.7] that there exist an integer \( m > 0 \), a set \( F \) of positive measure, and a \( \mu \)-measurable operator-valued function \( T \) defined on \( F \) such that \( (\lambda, T(\lambda)) \in E(f, k, m) \) for each \( f \in \mathcal{F} \) and each \( k \). Extend \( T \) to all of \( \Delta \) by letting \( T(\lambda) = 0 \) if \( \lambda \notin F \), and set \( T = \int_{\Delta} \Theta T(\lambda) \mu(d\lambda) \). Then \( T \in B(H) \), and, if \( \mathfrak{A} \) has property \( P \), there are functions \( g_k \in \mathcal{E}(\mathfrak{A}) \) and an operator \( A' \in \mathcal{A}' \) such that \( g_k \cdot T \mapsto A' \) weakly. By [8, Lemma 1.7] and [2, Corollary III.6.13] we may assume \( (g_k \cdot T \in \mathcal{Z}' \) for each \( k \)) that \( (g_k \cdot T)(\lambda) - A'(\lambda) \to 0 \) weakly \( \mu \)-a.e. In particular this must be true for \( \mu \)-a.e. \( \lambda \) in \( F \), which, by our remarks beginning the proof of Theorem 4, contradicts the fact that \( (\lambda, T(\lambda)) \in E(f, k, m) \) for each \( f \in \mathcal{F} \) and each \( k \) and all \( \lambda \in F \). Hence \( \mathfrak{A} \) does not have property \( P \).
To prove the converse, it suffices by Lemma 3 to show that if \( \mu(P') = 0 \) then \( \alpha \) has property CP. We may restrict our attention to \( T \in Z'_1 \), with \( T = \int \alpha \otimes T(\lambda) \mu(\alpha) \). By [7, p. 228], [8, Lemma 1.7], [2, Corollary III.6.13], and the remark following [8, Lemma 2.2] we may assume that \( \Lambda \) is compact and that \( T(\lambda) \) is strong-* continuous in \( \lambda \). Moreover, since \( \mu(P') = 0 \), we may assume that \( \alpha(\lambda) \) has property P for every \( \lambda \) (see remark following [7, Corollary 1.5.10]). Thus given any integer \( m \) and any \( \lambda \in \Lambda \), there are \( f \in \mathcal{F} \) and \( A'_i \) such that

\[
W((f \cdot T)(\lambda) - A'_i(\lambda)) < 1/m.
\]

By continuity each such inequality holds on an open set, and by compactness there is a finite cover by such sets. Hence there are disjoint \( \mu \)-measurable sets \( F_i \) such that \( \Lambda = \bigcup_{i=1}^n F_i \) and such that for each \( F_i \) there are \( f_i \in \mathcal{F} \) and \( A'_i \) for which

\[
W((f_i \cdot T)\lambda - A'_i(\lambda)) < 1/m \quad \text{for } \lambda \in F_i.
\]

We now show that there are \( h \in \mathcal{E}(\alpha) \) and \( A' \in \mathcal{A}' \) for which

\[
W((h \cdot T)(\lambda) - A'(\lambda)) < 1/m \quad \text{for every } \lambda.
\]

It then follows that \( \alpha \) has property CP, and our theorem is proved.

For convenience of notation, assume \( \Lambda = F \cup G \), with \( f, g \) and \( A'_i, A'_i' \) as above. Let \( V_1, \ldots, V_n \) be the support of \( f \), and let \( W_1, \ldots, W_m \) be the support of \( g \). Define unitaries \( U_{i,j} \) by \( U_{i,j}(\lambda) = V_i(\lambda) \) if \( \lambda \in F \) and \( W_j(\lambda) \) if \( \lambda \in G \). Define \( h \) with support on the \( U_{i,j} \) by \( h(U_{i,j}) = f(V_i)g(W_j) \). Define \( A' \in \mathcal{A}' \) by \( A'(\lambda) = A'_i(\lambda), \lambda \in F \) and \( A'(\lambda) = A'_i'(\lambda), \lambda \in G \). Then it is clear that \( h \) and \( A' \) give the desired result. Clearly the construction does not depend on the number of sets \( F_i \), and our theorem is proved. Q.E.D.

**Corollary 6.** If \( \alpha \) is of type I, then \( \alpha \) has property P.

**Proof.** It suffices to note that each type I factor on \( K \) has property P. This is clear for \( B(K) \) and for finite-dimensional factors, and the general result follows, since property P is a *-isomorphism invariant [3], from the known structure of type I factors. Q.E.D.

Schwartz introduced property P in [5], [6] as a property of hyperfinite factors. Since Powers has constructed a continuum of hyperfinite type III factors [4] and Ching has constructed a continuum of type III factors not having property P [1] and therefore not hyperfinite, the following corollary has some interest.
Corollary 7. Let $\mathcal{A}$ be a $W^*$ algebra on separable Hilbert space $H$. Then there is a projection $E \in \mathcal{Z}$ such that $\mathcal{A}_E$ has property $P$ and such that $\mathcal{Z} = \mathcal{A}_{I-E}$ contains no central projection $F \neq 0$ for which $\mathcal{A}_F$ has property $P$.

Proof. Let $E$ be the projection induced by the characteristic function of $\varnothing$. Q.E.D.

Bibliography