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ENERGY-FINITE SOLUTIONS OF $\Delta u = Pu$ AND DIRICHLET MAPPINGS

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ABSTRACT. Let R, S be noncompact Riemannian m-manifolds and let $T:R\to S$ be a Dirichlet mapping. Consider a nonnegative locally bounded measurable m-form P on R and set $Q=T_*^{-1}P$, the pull-back of P under T^{-1}. Denote by $PE(R)$ ($QE(S)$ resp.) the space of energy-finite solutions of $\Delta u=Pu$ on R ($\Delta u=Qu$ on S resp.). The spaces $PE(R)$ and $QE(S)$ are isomorphic, the isomorphism being bicontinuous with respect to the energy norms and preserves the sup norm of bounded solutions.

Let R, S be noncompact Riemannian m-manifolds and let $T:R\to S$ be a Dirichlet mapping, i.e. T is a homeomorphism and T is quasiconformal (quasi-isometric) if $m=2$ ($m\geqq 3$). Nakai [3], [7] has shown that such a T exists if and only if the Royden algebras of R and S are isomorphic. Thus the existence of a Dirichlet mapping of R onto S implies that the spaces of Dirichlet-finite harmonic functions on R and S are isomorphic (cf. [4], [5]).

In [1] the first step towards generalizing this result to energy-finite solutions of the equation $\Delta u=Pu$ ($P\geqq 0$) was taken. In particular the following canonical method of associating an m-form Q on S to an m-form P on R was introduced:

$$Q = T_*^{-1}P, \tag{1}$$

where T_*^{-1} denotes the pull-back mapping induced by T^{-1}. This definition gains its significance from two recent works. First, Nakai in [6] has shown that integrals are transformed according to the usual change of variable formula under Dirichlet mappings, which are far from C^1 in general. And second, the paper of Glasner-Katz [2] characterizes the space $PE(R)$ of solutions of $\Delta u=Pu$ on R with finite energy integrals in terms of the integral of P in neighborhoods of points of the Royden boundary of R. The main result in [1] is that either both dim $PE(R)$ and dim $QE(S)$ are infinite or they are equal.

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In this paper we establish the full generalization: the spaces $PE(R)$ and $QE(S)$ are isomorphic; the isomorphism is bicontinuous with respect to the energy norms and is an isometry with respect to the sup norms on the subspaces of bounded solutions.

We begin by fixing our terminology and stating some previously established results. Here a *Riemannian m-manifold* R will be taken to mean a C^1, orientable, connected, separable, noncompact m-manifold which is provided with a symmetric tensor (g_{ij}) that is merely Borel measurable. Moreover, we assume that there exists a covering of R by parametric balls B and a constant κ such that

$$\kappa^{-1}|\xi|^2 \leqq \xi(g_{ij}(x))\xi^t \leqq \kappa|\xi|^2,$$

for every vector $\xi\in E^m$ and every $x\in B$. The Royden algebra $M(R)$ associated with R is defined to be the set of bounded Tonelli functions (i.e. continuous functions with locally square integrable weak partial derivatives) with finite Dirichlet integrals $D_R(f)=\int_R df\wedge *df$ over R. The Royden compactification R^* of R can be viewed as the maximal ideal space of $M(R)$ (cf. [8]). Let Δ_R denote the harmonic part of $R^*\backslash R$.

Consider a measurable locally bounded m-form P on R which is (essentially) nonnegative and positive on a set of positive measure. We denote by $\tilde{E}(R)$ the Tonelli functions on R with finite energy integrals $E_R(f)=D_R(f)+\int_R f^2P$ over R and by $E(R)$ the bounded functions in $\tilde{E}(R)$. It is easily seen that the functions in $\tilde{E}(R)$ have continuous extended real-valued extensions to R^*. Throughout we shall use the same symbol to denote a function on R and its extension to R^*. The notation $PE(R)$ is used for the space of (weak) solutions of $d^*du=Pu$ on R with $u\in\tilde{E}(R)$. Also set

$$\overset{P}{\Delta}_R = \left\{ p\in\Delta_R \,\middle|\, p \text{ has a neighborhood } U \text{ in } R^* \text{ with } \int_{U\cap R} P<\infty \right\}.$$

We can state the (cf. [1, Theorem 8.2, Corollary 9.2])

LEMMA. *There exists a linear mapping* $\pi_P:\tilde{E}(R)\to PE(R)$ *such that*

$$E_R(\pi_Pf)\leqq E_R(f) \tag{2}$$

and

$$\pi_Pf\,|\,\overset{P}{\Delta}_R = f\,|\,\overset{P}{\Delta}_R. \tag{3}$$

For any $u\in PE(R)$

$$|u| \leqq \sup_{\overset{P}{\Delta}_R}|u|. \tag{4}$$

In particular,

$$\sup_R |\pi_P f| = \sup_{\Delta_R^P} |f|. \tag{5}$$

Nakai [**3**], [**7**] has shown that the Dirichlet mapping T induces an isomorphism $\sigma: M(R) \to M(S)$ between the Royden algebras of R and S by defining $f^\sigma = f \circ T^{-1}$ for $f \in M(R)$. Moreover there exists a constant k such that

$$k^{-1} D_R(f) \leqq D_S(f^\sigma) \leqq k D_R(f) \tag{6}$$

for every $f \in M(R)$.

If T takes local coordinates x on R into y on S, then the m-form Q defined in (1) is given by $\psi(x(y)) |J_x(y)| dy^1 \wedge \cdots \wedge dy^m$, where $P = \psi(x) dx^1 \wedge \cdots \wedge dx^m$ and $J_x(y)$ is the Jacobian of T^{-1} in terms of x and y. Thus by Nakai's result [**6**, Theorem 10] we have $\int_R f^2 P = \int_S (f^\sigma)^2 Q$. This together with (6) gives

$$k^{-1} E_R(f) \leqq E_S(f^\sigma) \leqq k E_R(f) \tag{7}$$

for every $f \in E(R)$.

The isomorphism σ can be extended to a vector space isomorphism of $\tilde{E}(R)$ and $\tilde{E}(S)$. In fact given $f \in \tilde{E}(R)$ we set $f^\sigma = f \circ T^{-1}$. To see that $f^\sigma \in \tilde{E}(S)$ consider $f_n = \max(-n, \min(f, n)) \in E(R)$. Then $\{f_n\}$ is Cauchy in the energy norm and by virtue of (7) so is $\{f_n^\sigma\}$. Hence the sequence $\{f_n^\sigma\}$ converges to f^σ in the energy-norm as well as pointwise and (7) remains valid for $f \in \tilde{E}(R)$.

In a natural way σ can be used to extend the Dirichlet mapping T to a homeomorphism $T: R^* \to S^*$ of the Royden compactifications. For a given $p \in R^*$ simply define Tp to be the maximal ideal of $M(S)$ given by $g(Tp) = g^{\sigma^{-1}}(p)$ for every $g \in M(S)$. For this extension Nakai [**4**], [**5**] has shown that $T|\Delta_R$ is a homeomorphism of Δ_R onto Δ_S. In view of the change of variable formula, (1) and the definition of Δ_R^P, Δ_S^Q we have the

LEMMA. *$T|\Delta_R^P$ is a homeomorphism of Δ_R^P onto Δ_S^Q with*

$$g(Tp) = g^{\sigma^{-1}}(p) \qquad \text{for } g \in \tilde{E}(R). \tag{8}$$

We now establish our result.

THEOREM. *Let $\Sigma: PE(R) \to QE(S)$ be defined by $\Sigma u = \pi_Q(u^\sigma)$, for $u \in PE(R)$. Then Σ is an isomorphism with*

$$k^{-1} E_R(u) \leqq E_S(\Sigma u) \leqq k E_R(u) \tag{9}$$

and

$$\sup_R |u| = \sup_S |\Sigma u|. \tag{10}$$

For the proof we define $\Sigma': QE(S) \to PE(R)$ by $\Sigma' v = \pi_P(v^{\sigma^{-1}})$ and show that Σ' is the inverse of the linear transformation Σ. Let $p \in \Delta_R^P$. Then

$$\begin{aligned}(\Sigma'\Sigma u)(p) &= \pi_P((\Sigma u)^{\sigma^{-1}})(p) = (\Sigma u)^{\sigma^{-1}}(p) \\ &= \Sigma u(Tp) = \pi_Q(u^\sigma)(Tp) = u^\sigma(Tp) = u(p),\end{aligned}$$

by using (3), (8) and the definitions of Σ, Σ' repeatedly. Thus $\Sigma'\Sigma u$ and u agree on Δ_R^P and consequently (4) implies that they are identical. Hence $\Sigma'\Sigma$ is the identity on $PE(R)$ and by symmetry $\Sigma\Sigma'$ is the identity on $QE(S)$.

Note that $E_S(\Sigma u) \leqq E_S(u^\sigma)$ by (2) and thus (7) gives the second inequality of (9). The symmetric inequality for Σ' is $E_R(\Sigma'\Sigma u) \leqq kE_S(\Sigma u)$ which is the first inequality of (9). Finally we obtain

$$\sup_S |\Sigma u| = \sup_{\Delta_S^Q} |u^\sigma| = \sup_{\Delta_R^P} |u| = \sup_R |u|$$

by invoking (5), (8) and (4).

References

1. M. Glasner, *Dirichlet mappings of Riemannian manifolds and the equation* $\Delta u = Pu$, J. Differential Equations **9** (1971).

2. M. Glasner and R. Katz, *On the behavior of solutions of* $\Delta u = Pu$ *at the Royden boundary*, J. Analyse Math. **22** (1969), 343–354. MR **41** #1995.

3. M. Nakai, *Algebraic criterion on quasiconformal equivalence of Riemann surfaces*, Nagoya Math. J. **16** (1960), 157–184. MR **22** #1669.

4. ———, *Royden's map between Riemann surfaces*, Bull. Amer. Math. Soc. **72** (1966), 1003–1005. MR **33** #7528.

5. ———, *On parabolicity and Royden compactifications of Riemannian manifolds*, Proc. Internat. Congress Functional Analysis, Tokyo, 1969.

6. ———, *Radon-Nikodym densities and Jacobians*, Pacific J. Math. (to appear).

7. ———, *Roydan algebras and quasi-isometries of Riemannian manifolds*, Pacific J. Math. (to appear).

8. L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Springer-Verlag, Berlin, 1970.

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