A WHITEHEAD TYPE THEOREM

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Abstract. Let $\mathfrak{F}$ denote the Serre class of finite abelian groups. We consider, for example, conditions under which a map which induces an $\mathfrak{F}$-epimorphism in homotopy also induces an $\mathfrak{F}$-epimorphism in homology.

1. Introduction. Let $f: X \to Y$ be a map, $\mathcal{C}$ a Serre class of abelian groups. Modulo some technical assumptions on the spaces $X$ and $Y$ or the class $\mathcal{C}$, the Whitehead theorem states that $f$ induces a $\mathcal{C}$-isomorphism in homotopy in each dimension if and only if it induces a $\mathcal{C}$-isomorphism in homology in each dimension. We are concerned here with finding conditions under which the word “isomorphism” can be replaced by “epimorphism” or “monomorphism” for the class $\mathfrak{F}$ of finite abelian groups. For example, we show that if a map $g$ from an $H$-space $Y$ to a 1-connected finite CW-complex $X$ induces an $\mathfrak{F}$-epimorphism in homotopy, then it induces an $\mathfrak{F}$-epimorphism in homology and $X$ is an $\mathfrak{F}$-space mod $\mathfrak{F}$. As a special case we recover a result of [4]: If $X$ is a 1-connected finite CW-complex and a $G$-space mod $\mathfrak{F}$ (i.e. the evaluation map $\omega: (X^X, 1) \to (X, *)$ induces an $\mathfrak{F}$-epimorphism in homotopy), then $X$ is an $H$-space mod $\mathfrak{F}$. (The converse is also true.) Moreover, $\omega$ induces an $\mathfrak{F}$-epimorphism in homology. The proof given here is much simpler than that given in [4].

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All spaces are assumed to have the based homotopy type of a CW-complex and all maps and homotopies are to preserve base points. We will frequently not distinguish between a map and its homotopy class. The symbol “$\mathfrak{F}$” will be used to denote Hurewicz homomorphisms. We assume that the reader is familiar with [1] and [2].

2. The result. Let $X$ be a 1-connected finite CW-complex, $Y$ be a 1-connected space with $H_m(Y)$ finitely generated for all $m$ and let $f: X \to Y, g: Y \to X$ be maps.

Theorem 1. Suppose that $Y$ is an $H$-space.

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(i) If \( f_* : \pi_m(X) \to \pi_m(Y) \) is an \( H \)-monomorphism for all \( m \), then so is 
\[ f_* : H_m(X) \to H_m(Y). \]
(ii) If \( g_* : \pi_m(Y) \to \pi_m(X) \) is an \( H \)-epimorphism for all \( m \), then so is 
\[ g_* : H_m(Y) \to H_m(X). \]
Moreover, \( X \), in each case, is an \( H \)-space mod \( H \).

**Theorem 2.** Suppose that \( Y \) is an \( H' \)-space.
(i) If \( f_* : H_m(X) \to H_m(Y) \) is an \( H \)-monomorphism for all \( m \), then so is 
\[ f_* : \pi_m(X) \to \pi_m(Y). \]
(ii) If \( g_* : H_m(Y) \to H_m(X) \) is an \( H \)-epimorphism for all \( m \), then so is 
\[ g_* : \pi_m(Y) \to \pi_m(X). \]
Moreover, \( X \), in each case, is an \( H' \)-space mod \( H \).

**Remark.** The 1-connectedness assumption on \( F \) is needed only for
Theorem 2 (ii) and neither assumption on \( Y \) is needed for Theorem 1
(ii).

We will need the following result, the proof of which depends only
on the universal coefficient theorem and the representability of
cohomology.

**Lemma 1.** Let \( B \) be a space for which \( H_m(B) \) is finitely generated and
let \( \beta \in \pi_m(B) \). Then \( f(\beta) \neq 0 \) if and only if there is a map \( h : B \to K(\pi, m) \)
(= an Eilenberg-Mac Lane space) such that \( h\beta \) is not homotopic to a
constant. (We may take \( \pi = Z_p \) or \( Z \) depending on whether \( f(\beta) \) has
finite or infinite order.)

**Lemma 2.** Let \( a : A \to B \) be a map from an \( H \)-space \( A \) to a finite CW-
complex \( B \). If \( a_* : \pi_{2n}(A) \to \pi_{2n}(B) \) is an \( H \)-epimorphism, then \( f(\pi_{2n}(B)) \in \mathcal{F} \).

**Proof.** It suffices to show that \( f(\beta) \) has finite order for each
\( \beta \in \pi_{2n}(B) \). In order to obtain a contradiction, assume that there is a
\( \beta \in \pi_{2n}(B) \) such that \( f(\beta) \) has infinite order. By Lemma 1, there is a
map \( h : B \to K(Z, 2n) \) such that \( h\beta \) is not homotopic to a constant.
Since \( a_* : \pi_{2n}(A) \to \pi_{2n}(B) \) is an \( H \)-epimorphism there is an \( \alpha \in \pi_{2n}(A) \)
such that \( a_*(\alpha) = r\beta \) where \( r \) is some nonzero integer. Now, if \( p : \Sigma A \to A \)
is a retraction map (\( A \) is an \( H \)-space), then \( \text{hop} \circ \Omega \Sigma \alpha : \Sigma S^{2n} \to K(Z, 2n) \) is a nontrivial map which factors through a finite complex.
This is clearly impossible (consider the ring structure of \( H^*(\Sigma S^{2n}) \)) and the lemma is proved.

**Corollary 1.** If \( B \) is \((2n - 1)\)-connected, then \( \pi_{2n}(B) \in \mathcal{F} \).

**Proof of Theorem 1.** (i) Since the homotopy suspension homomorphism
for an \( H \)-space is a monomorphism in all dimensions, it follows that the suspension homomorphism \( i_* : \pi_m(X) \to \pi_m(\Omega X) \) (\( i \) is the inclusion map) is an \( H \)-monomorphism for all \( m \) and therefore that \( X \) is an \( H \)-space mod \( H \). Let \( h : S \to X \) be a weak \( H \)-equivalence,
where $S$ is a finite product of odd dimensional spheres. Since the Hurewicz homomorphism $h : \pi_m(Y) \to H_m(Y)$ is an $\mathfrak{g}$-monomorphism it follows from Lemma 1 that $fh$ induces an $\mathfrak{g}$-epimorphism in cohomology and hence an $\mathfrak{g}$-monomorphism in homology. Thus $f$ induces an $\mathfrak{g}$-monomorphism in homology.

(ii) We first show, by induction, that $\pi_{2n}(X)\in\mathfrak{g}$ for all $n$. For $n=1$, this follows from Corollary 1. Assume that $\pi_{2n}(X)\in\mathfrak{g}$ for $2n<N$, $N$ odd. Since $g_* : \pi_m(Y)\to\pi_m(X)$ is an $\mathfrak{g}$-epimorphism for all $m$, we can use the multiplication on $Y$ to obtain a map $h_N : S\to Y$ such that $gh_N$ induces an $\mathfrak{g}$-isomorphism in homotopy in dimensions $\leq N$, where $S$ is a finite product of odd dimensional spheres $S^n$, $3\leq n\leq N$. We can assume that $gh_N$ is an inclusion map. Then $\pi_m(X, S)\in\mathfrak{g}$ for all $m\leq N$; by the Hurewicz theorem, $h : \pi_{N+1}(X, S)\to H_{N+1}(X, S)$ is an $\mathfrak{g}$-isomorphism. Since $\pi_{N+1}(S)\in\mathfrak{g}$, it follows that $h : \pi_{N+1}(X)\to H_{N+1}(X)$ is an $\mathfrak{g}$-monomorphism and so, by Lemma 2, $\pi_{2n}(X)\in\mathfrak{g}$ for all $n$. It is now a simple matter to show that for $N\geq \dim X$, $gh_N$ is a weak $\mathfrak{g}$-equivalence. Therefore $X$ is an $H$-space mod $\mathfrak{g}$ and $g_* : H_m(Y)\to H_m(X)$ is an $\mathfrak{g}$-epimorphism for all $m$.

Proof of Theorem 2. (i) Since $f_* : H_m(X)\to H_m(Y)$ is an $\mathfrak{g}$-monomorphism for all $m$, $f^* : H^m(Y)\to H^m(X)$ is an $\mathfrak{g}$-epimorphism for all $m$. Let $\{\beta_i\}$ be a basis for the free part of $H^*(X)$ and let $\{\gamma_i\} \subset H^*(Y)$ be chosen so that $f^* (\gamma_i) = i_i \beta_i$ for some nonzero integer $i_i$. Let $r > \dim X$ be arbitrary, $\gamma_i' = \gamma_i| Y^r$, where $Y^r$ is the $r$-skeleton of $Y$.

Since $\gamma_i \cup \gamma_i' = 0$ ($Y$ is an $H'$-space), $\gamma_i' \cup \gamma_i' = 0$ and [3] there is a map $h_i : Y^r \to S^{n_i}$, $n_i = \dim \gamma_i$, which maps the fundamental class of $S^{n_i}$ to some nonzero multiple of $\gamma_i$. Making use of the fact that $Y$ is an $H'$-space we obtain a map $h : Y^r \to VS^{n_i}$ (as in [3]) such that $hf$ is a weak $\mathfrak{g}$-equivalence (by the cellular approximation theorem we can assume $f(X) \subset Y^r$). Therefore $X$ is an $H'$-space mod $\mathfrak{g}$ and $f_* : \pi_m(X) \to \pi_m(Y)$ is an $\mathfrak{g}$-monomorphism for $m < r$. Since $r$ was arbitrary the result follows.

(ii) Since the Hurewicz homomorphism for an $H'$-space is an $\mathfrak{g}$-epimorphism in all dimensions, it follows that $h : \pi_m(X)\to H_m(X)$ is an $\mathfrak{g}$-epimorphism for all $m$ and hence that $X$ is an $H'$-space mod $\mathfrak{g}$. Moreover, it is clear that there is a map $h : VS^{n_i}\to Y$ such that $fh$ is a weak $\mathfrak{g}$-equivalence and the result follows.

Remark. In contrast to the Whitehead theorem, the converse of each assertion in Theorems 1 and 2 is false. Counterexamples are given as follows:

1(i). The inclusion map $S^{2n+1}\to\Omega S S^{2n}$.

1(ii). The quotient map $S^n \times S^n/L S^n = S^{2n}, n = 3$ or 7.
2(i). The Whitehead product map $S^{m+n-1} \to S^m \vee S^n$, $m + n$ even, $m, n \geq 2$.

2(ii). The inclusion map $S^m \vee S^n \to S^m \times S^n$.

**Bibliography**


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