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A WHITEHEAD TYPE THEOREM

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ABSTRACT. Let $\mathfrak{F}$ denote the Serre class of finite abelian groups. We consider, for example, conditions under which a map which induces an $\mathfrak{F}$-epimorphism in homotopy also induces an $\mathfrak{F}$-epimorphism in homology.

1. **Introduction.** Let $f: X \to Y$ be a map, $\mathfrak{C}$ a Serre class of abelian groups. Modulo some technical assumptions on the spaces X and Y or the class $\mathfrak{C}$, the Whitehead theorem states that f induces a $\mathfrak{C}$-isomorphism in homotopy in each dimension if and only if it induces a $\mathfrak{C}$-isomorphism in homology in each dimension. We are concerned here with finding conditions under which the word "isomorphism" can be replaced by "epimorphism" or "monomorphism" for the class $\mathfrak{F}$ of finite abelian groups. For example, we show that if a map g from an H-space Y to a 1-connected finite CW-complex X induces an $\mathfrak{F}$-epimorphism in homotopy, then it induces an $\mathfrak{F}$-epimorphism in homology and X is an H-space mod $\mathfrak{F}$. As a special case we recover a result of [4]: If X is a 1-connected finite CW-complex and a G-space mod $\mathfrak{F}$ (i.e. the evaluation map $\omega: (X^X, 1) \to (X, *)$ induces an $\mathfrak{F}$-epimorphism in homotopy), then X is an H-space mod $\mathfrak{F}$. (The converse is also true.) Moreover, ω induces an $\mathfrak{F}$-epimorphism in homology. The proof given here is much simpler than that given in [4].

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All spaces are assumed to have the based homotopy type of a CW-complex and all maps and homotopies are to preserve base points. We will frequently not distinguish between a map and its homotopy class. The symbol "$\mathfrak{h}$" will be used to denote Hurewicz homomorphisms. We assume that the reader is familiar with [1] and [2].

2. **The result.** Let X be a 1-connected finite CW-complex, Y be a 1-connected space with $H_m(Y)$ finitely generated for all m and let $f: X \to Y$, $g: Y \to X$ be maps.

THEOREM 1. *Suppose that Y is an H-space.*

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(i) *If* $f_*:\pi_m(X)\to\pi_m(Y)$ *is an* $\mathfrak{F}$*-monomorphism for all* m, *then so is* $f_*:H_m(X)\to H_m(Y)$.

(ii) *If* $g_*:\pi_m(Y)\to\pi_m(X)$ *is an* $\mathfrak{F}$*-epimorphism for all* m, *then so is* $g_*:H_m(Y)\to H_m(X)$. *Moreover*, X, *in each case, is an* H*-space* mod $\mathfrak{F}$.

THEOREM 2. *Suppose that* Y *is an* H'*-space.*

(i) *If* $f_*:H_m(X)\to H_m(Y)$ *is an* $\mathfrak{F}$*-monomorphism for all* m, *then so is* $f_*:\pi_m(X)\to\pi_m(Y)$.

(ii) *If* $g_*:H_m(Y)\to H_m(X)$ *is an* $\mathfrak{F}$*-epimorphism for all* m, *then so is* $g_*:\pi_m(Y)\to\pi_m(X)$. *Moreover*, X, *in each case, is an* H'*-space* mod $\mathfrak{F}$.

REMARK. The 1-connectedness assumption on Y is needed only for Theorem 2 (ii) and neither assumption on Y is needed for Theorem 1 (ii).

We will need the following result, the proof of which depends only on the universal coefficient theorem and the representability of cohomology.

LEMMA 1. *Let* B *be a space for which* $H_m(B)$ *is finitely generated and let* $\beta\in\pi_m(B)$. *Then* $\mathfrak{h}(\beta)\neq 0$ *if and only if there is a map* $h:B\to K(\pi, m)$ (*=an Eilenberg-Mac Lane space*) *such that* $h\beta$ *is not homotopic to a constant.* (*We may take* $\pi=Z_{p^r}$ *or* Z *depending on whether* $\mathfrak{h}(\beta)$ *has finite or infinite order.*)

LEMMA 2. *Let* $a:A\to B$ *be a map from an* H*-space* A *to a finite* CW-*complex* B. *If* $a_*:\pi_{2n}(A)\to\pi_{2n}(B)$ *is an* $\mathfrak{F}$*-epimorphism, then* $\mathfrak{h}(\pi_{2n}(B))\in\mathfrak{F}$.

PROOF. It suffices to show that $\mathfrak{h}(\beta)$ has finite order for each $\beta\in\pi_{2n}(B)$. In order to obtain a contradiction, assume that there is a $\beta\in\pi_{2n}(B)$ such that $\mathfrak{h}(\beta)$ has infinite order. By Lemma 1, there is a map $h:B\to K(Z, 2n)$ such that $h\beta$ is not homotopic to a constant. Since $a_*:\pi_{2n}(A)\to\pi_{2n}(B)$ is an $\mathfrak{F}$-epimorphism there is an $\alpha\in\pi_{2n}(A)$ such that $a_*(\alpha)=r\beta$ where r is some nonzero integer. Now, if $\rho:\Omega\Sigma A\to A$ is a retraction map (A is an H-space), then $ha\rho\circ\Omega\Sigma\alpha:\Omega\Sigma S^{2n}\to K(Z, 2n)$ is a nontrivial map which factors through a finite complex. This is clearly impossible (consider the ring structure of $H^*(\Omega\Sigma S^{2n})$) and the lemma is proved.

COROLLARY 1. *If* B *is* $(2n-1)$*-connected, then* $\pi_{2n}(B)\in\mathfrak{F}$.

PROOF OF THEOREM 1. (i) Since the homotopy suspension homomorphism for an H-space is a monomorphism in all dimensions, it follows that the suspension homomorphism $i_*:\pi_m(X)\to\pi_m(\Omega\Sigma X)$ (i is the inclusion map) is an $\mathfrak{F}$-monomorphism for all m and therefore that X is an H-space mod $\mathfrak{F}$. Let $h:S\to X$ be a weak $\mathfrak{F}$-equivalence,

where S is a finite product of odd dimensional spheres. Since the Hurewicz homomorphism $\mathfrak{h}:\pi_m(Y)\to H_m(Y)$ is an $\mathfrak{F}$-monomorphism it follows from Lemma 1 that fh induces an $\mathfrak{F}$-epimorphism in cohomology and hence an $\mathfrak{F}$-monomorphism in homology. Thus f induces an $\mathfrak{F}$-monomorphism in homology.

(ii) We first show, by induction, that $\pi_{2n}(X)\in\mathfrak{F}$ for all n. For $n=1$, this follows from Corollary 1. Assume that $\pi_{2n}(X)\in\mathfrak{F}$ for $2n<N$, N odd. Since $g_*:\pi_m(Y)\to\pi_m(X)$ is an $\mathfrak{F}$-epimorphism for all m, we can use the multiplication on Y to obtain a map $h_N:S\to Y$ such that gh_N induces an $\mathfrak{F}$-isomorphism in homotopy in dimensions $\leqq N$, where S is a finite product of odd dimensional spheres S^{n_i}, $3\leqq n_i\leqq N$. We can assume that gh_N is an inclusion map. Then $\pi_m(X, S)\in\mathfrak{F}$ for all $m\leqq N$; by the Hurewicz theorem, $\mathfrak{h}:\pi_{N+1}(X, S)\to H_{N+1}(X, S)$ is an $\mathfrak{F}$-isomorphism. Since $\pi_{N+1}(S)\in\mathfrak{F}$, it follows that $\mathfrak{h}:\pi_{N+1}(X)\to H_{N+1}(X)$ is an $\mathfrak{F}$-monomorphism and so, by Lemma 2, $\pi_{N+1}(X)\in\mathfrak{F}$. Thus $\pi_{2n}(X)\in\mathfrak{F}$ for all n. It is now a simple matter to show that for $N\geqq\dim X$, gh_N is a weak $\mathfrak{F}$-equivalence. Therefore X is an H-space mod $\mathfrak{F}$ and $g_*:H_m(Y)\to H_m(X)$ is an $\mathfrak{F}$-epimorphism for all m.

Proof of Theorem 2. (i) Since $f_*:H_m(X)\to H_m(Y)$ is an $\mathfrak{F}$-monomorphism for all m, $f^*:H^m(Y)\to H^m(X)$ is an $\mathfrak{F}$-epimorphism for all m. Let $\{\beta_i\}$ be a basis for the free part of $H^*(X)$ and let $\{\gamma_i\}\subset H^*(Y)$ be chosen so that $f^*(\gamma_i)=t_i\beta_i$ for some nonzero integer t_i. Let $r>\dim X$ be arbitrary, $\gamma_i'=\gamma_i|\,Y^r$, where Y^r is the r-skeleton of Y.

Since $\gamma_i\cup\gamma_i=0$ (Y is an H'-space), $\gamma_i'\cup\gamma_i'=0$ and [3] there is a map $h_i: Y^r\to S^{n_i}$, $n_i=\dim\gamma_i$, which maps the fundamental class of S^{n_i} to some nonzero multiple of γ_i'. Making use of the fact that Y is an H'-space we obtain a map $h: Y^r\to\bigvee S^{n_i}$ (as in [3]) such that hf is a weak $\mathfrak{F}$-equivalence (by the cellular approximation theorem we can assume $f(X)\subset Y^r$). Therefore X is an H'-space mod $\mathfrak{F}$ and $f_*:\pi_m(X)\to\pi_m(Y)$ is an $\mathfrak{F}$-monomorphism for $m<r$. Since r was arbitrary the result follows.

(ii) Since the Hurewicz homomorphism for an H'-space is an $\mathfrak{F}$-epimorphism in all dimensions, it follows that $\mathfrak{h}:\pi_m(X)\to H_m(X)$ is an $\mathfrak{F}$-epimorphism for all m and hence that X is an H'-space mod $\mathfrak{F}$. Moreover, it is clear that there is a map $h:\bigvee S^{n_i}\to Y$ such that fh is a weak $\mathfrak{F}$-equivalence and the result follows.

Remark. In contrast to the Whitehead theorem, the converse of each assertion in Theorems 1 and 2 is false. Counterexamples are given as follows:

1(i). The inclusion map $S^{2n}\to\Omega\Sigma S^{2n}$.

1(ii). The quotient map $S^n\times S^n\to S^n\wedge S^n=S^{2n}$, $n=3$ or 7.

2(i). The Whitehead product map $S^{m+n-1} \to S^m \vee S^n$, $m+n$ even, $m, n \geqq 2$.

2(ii). The inclusion map $S^m \vee S^n \to S^m \times S^n$.

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